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Developing geometric thinking

A developmental series of classroom activities for Gr. 1 - 9

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Developing Geometric Thinking through Activities That Begin with Play

Pierre M. van Hiele

For children, geometry begins with play. Rich and stimulating instruction in geometry can be provided through playful activities with mosaics, such as pattern blocks or design tiles, with puzzles like tangrams, or with the special seven-piece mosaic shown in **figure 1.** Teachers might ask, How can children use mosaics, and what geometry do they learn? Before addressing these questions and exploring the potential of the mosaic puzzle for teaching geometry, I note some misconceptions in the teaching of mathematics and present some of my ideas about levels of thinking in geometry.

Pierre M. van Hiele, a lifelong resident of the Netherlands, is a former Montessori teacher and the author of a curriculum series that features a rich array of geometry activities. He is also world renowned for his work on levels of thinking as they relate to geometry and discusses them in this article.

During a 1987 visit to Brooklyn College, van Hiele was introduced to this mosaic puzzle. Since then, he has been fascinated with this puzzle and with the many ways that it can be used to teach geometry. Although he has lectured on activities involving the mosaic puzzle, this is the only article of which he is aware that discusses ways of using it to teach geometry concepts.

This is the first article by van Hiele to be published in a journal of the National Council of Teachers of Mathematics. For this reason, the Editorial Panel of Teaching Children Mathematics *is especially grateful to have his article appear in the journal's 1999 Focus Issue on "Geometry and Geometric Thinking." The Panel also wishes to acknowledge the work of our colleague, David Fuys, who helped van Hiele prepare the final draft of his manuscript.—Charles Geer, for the Editorial Panel.*

Misunderstandings in Teaching Mathematics

The teaching of school mathematics—geometry and arithmetic—has been a source of many misunderstandings. Secondary school geometry was for a long time based on the formal axiomatic geometry that Euclid created more than 2000 years ago. His logical construction of geometry with its axioms, definitions, theorems, and proofs was—for its time—an admirable scientific achievement. School geometry that is presented in a similar axiomatic fashion assumes that students think on a formal deductive level. However, that is usually not the case, and they lack prerequisite understandings about geometry. This lack creates a gap between their level of thinking and that required for the geometry that they are expected to learn.

A similar misunderstanding is seen in the teaching of arithmetic in elementary school. As had been done by Euclid in geometry, mathematicians developed axiomatic constructions for arithmetic, which subsequently affected the arithmetic taught in schools. In the 1950s, Piaget and I took a stand against this misunderstanding. However, it did not help, for just then, set theory was established as the foundation for number, and school arithmetic based on sets was implemented worldwide in what was commonly called the "new math." For several years, this misconception dominated school mathematics, and the end came only after negative results were reported. Piaget's point of view, which I support affectionately, was that "giving no education is better than giving it at the wrong time." We must provide teaching that is appropriate to the level of children's thinking.

Levels of Geometric Thinking

At what level should teaching begin? The answer, of course, depends on the students' level of thinking. I begin to explain what I mean by levels of thinking by sharing a conversation that two of my daughters, eight and nine years old at the time, had about thinking. Their question was, If you are awake, are you then busy with thinking? "No," one said. "I can walk in the woods and see the trees and all the other beautiful things, but I do not think I see the trees. I see ferns, and I see them without thinking." The other said, "Then you have been thinking, or you knew you were in the woods and that you saw trees, but only you did not use words."

I judged this controversy important and asked the opinion of Hans Freudenthal, a prominent Dutch mathematician and educator. His conclusion was clear: Thinking without words is not thinking. In *Structure and Insight* (van Hiele 1986), I expressed this point of view, and psychologists in the United States were not happy with it. They were right: If nonverbal thinking does not belong to real thinking, then even if we are awake, we do not think most of the time.

Nonverbal thinking is of special importance; all rational thinking has its roots in nonverbal thinking, and many decisions are made with only that kind of thought. We observe some things without having any words for them. We recognize the faces of familiar persons without being able to use words to describe their faces. In my levels of geometric thinking, the "lowest" is the **visual level,** which begins with nonverbal thinking. At the visual level of thinking, figures are judged by their appearance. We say, "It is a square. I know that it is one because I see it is." Children might say, "It is a rectangle because it looks like a box."

At the next level, **the descriptive level,** figures are the bearers of their properties. A figure is no longer judged because "it looks like one" but rather because it has certain properties. For example, an equilateral triangle has such properties as three sides; all sides equal; three equal angles; and symmetry, both about a line and rotational. At this level, language is important for describing shapes. However, at the descriptive level, properties are not yet logically ordered, so a triangle with equal sides is not necessarily one with equal angles.

At the next level, **the informal deduction level,** properties are logically ordered. They are deduced from one another; one property precedes or follows from another property. Students use properties that they already know to formulate definitions, for example, for squares, rectangles, and equilateral triangles, and use them to justify relationships, such as explaining why all squares are rectangles or why the sum of the angle measures of the angles of any triangle must be 180. However, at this level, the intrinsic meaning of deduction, that is, the role of axioms, definitions, theorems, and their converses, is not understood. My experience as a teacher of geometry convinces me that all too often, students have not yet achieved this level of informal deduction. Consequently, they are not successful in their study of the kind of geometry that Euclid created, which involves formal deduction. See van Hiele (1997) and Fuys, Geddes, and Tischler (1988) for further information about the levels.

How do students develop such thinking? I believe that development is more dependent on instruction than on age or biological maturation and that types of instructional experiences can foster, or impede, development. As I discuss at the end of this article, instruction intended to foster development from one level to the next should include sequences of activities, beginning with an exploratory phase, gradually building concepts and related language, and culminating in summary activities that help students integrate what they have learned into what they already know. The following activities illustrate this type of sequence for developing thinking at the visual level and for supporting a transition to the descriptive level.

Beginning Geometry and the Mosaic Puzzle

Join me now in using the seven-piece mosaic (see **fig. 1**) in playful explorations that deal with certain shapes and their properties, symmetry, parallelism, and area. Before reading further, please make your own set of pieces to use in the activities, which can be adapted for children, depending on their prior geometric experiences. **Figure 1** can be reproduced on cardstock to make durable sets for yourself and your students. Pieces are numbered on their topsides for reference in directions and discussions of the activities.

Imagine that the large rectangle in **figure 1** has broken into seven pieces: an isosceles triangle (piece 1); an equilateral triangle (piece 2); two right triangles (pieces 5 and 6); and three quadrilaterals consisting of a rectangle (piece 3), a trapezoid (piece 7), and an isosceles trapezoid (piece 4). **Figure 2** shows how the large rectangle and its pieces fit on a grid pattern of equilateral triangles.

We begin by asking, What can we do with these pieces? Children respond to this open question by using their imaginations and playing with the pieces

FIGURE 1

The seven-piece mosaic puzzle—create a set of pieces to use as you read this article.

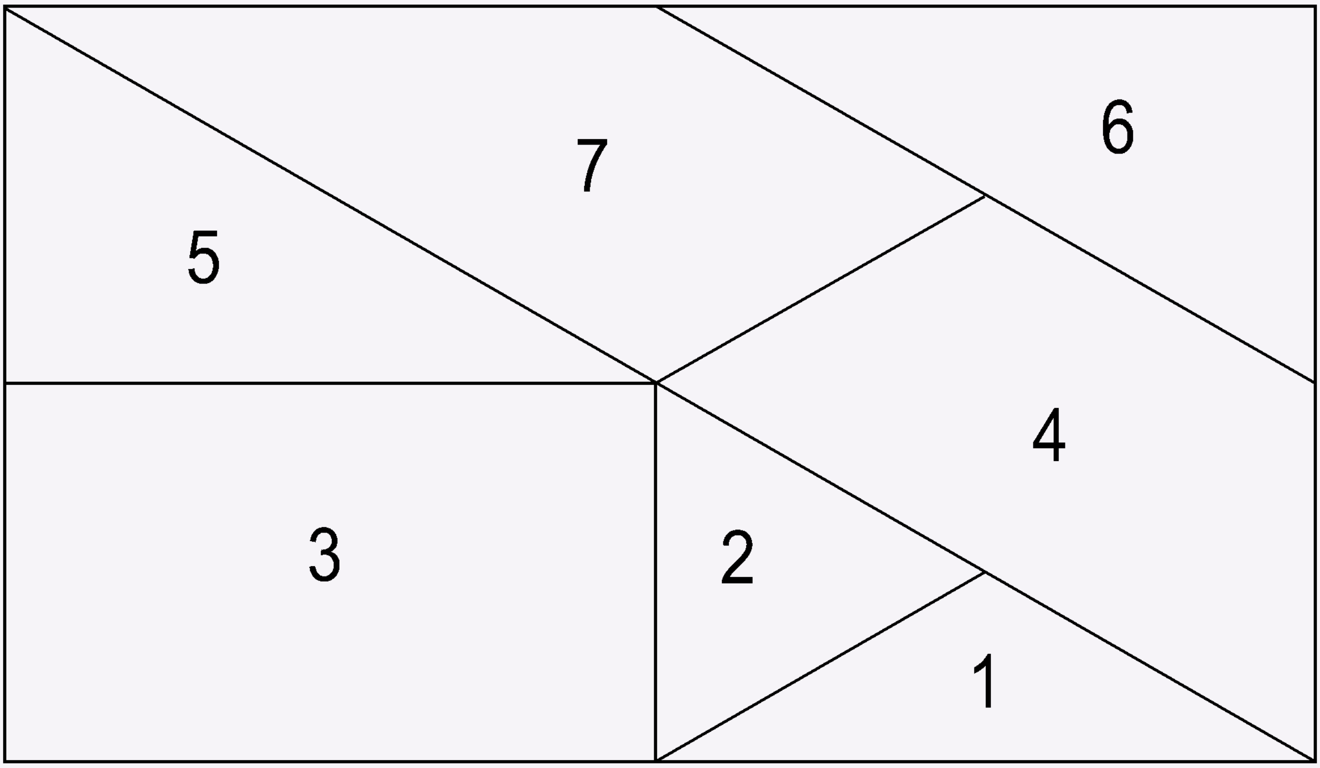


FIGURE 2

Equilateral-triangle grid

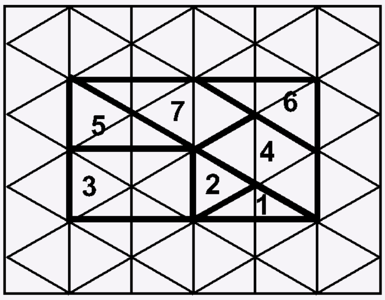
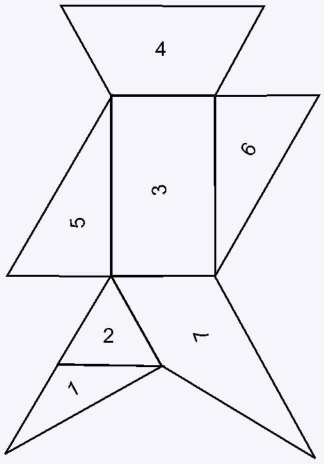


FIGURE 3

A person



to create whatever they wish—sometimes real-world objects like a person (see **fig. 3**) or a house (see **fig. 4**); sometimes other shapes, like piece 3, or abstract designs. Children should be given ample opportunity for free play and for sharing their creations. Such play gives teachers a chance to observe how children use the pieces and to assess informally how they think and talk about shapes.

In free play, children may have joined two pieces to make another piece, for example, using pieces 5 and 6 to make piece 3. We can ask them to find all the pieces that can be made from two others. Only pieces 1 and 2 cannot. Try this activity, and then find the one piece that can be made from three others. Children can place pieces directly on top of the piece that they want to make or form it next to the piece for easy visual comparison. To record their solutions, children should trace around a piece and then draw how they made it with the other pieces or show their method with colored markers on a triangle grid.

This activity leads children to notice that joining two pieces sometimes makes a shape that is not the same as one of the seven original pieces. They can investigate how many different shapes can be made with a pair of pieces, joining them by sides that match. With pieces 5 and 6, six shapes are possible, only one of which is the same as an original piece. Try these combinations, and then try the same activity with pieces 1 and 2.

New shapes are introduced by puzzles that require two or more pieces. The shape in **figure 5** can be made two ways. One uses pieces 2 and 4 number side up; the other uses pieces 2 and 4 flipped over with the number side down. Make the shape both ways. Can it be made with pieces 1 and 7? With pieces 1 and 7 flipped over? What other two pieces make this shape, and do they also work if they are flipped over?

Making the shape in different ways with two pieces may inspire children to ask, Can we make it with three pieces, too? Try pieces 1, 2, and 5, and then make it in a different way with these three pieces. Also, try pieces 1, 2, and 5 flipped over.

In solving puzzles like these, children work visually with angles that fit and sides that match.

They also notice that some pieces fit with either side up but that other pieces do not. Pieces 2 and 3 fit either side up; piece 7 does not, since flipping it changes its orientation and how it looks. Is piece 1 a flipable piece? Are pieces 4, 5, or 6?

Puzzle Cards and the Mosaic Puzzle

Next I present more complex puzzles. Directions can be given orally or by task cards, like those in **figure 6.** Read and try them. They illustrate how puzzles that are created with two pieces can have solutions that use other pieces. Think about what geometry they involve and the conversations that

FIGURE 4

A house

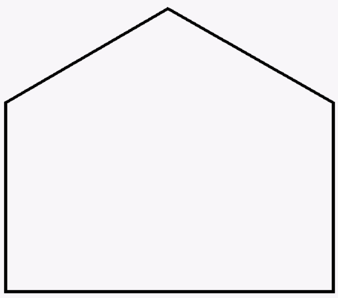
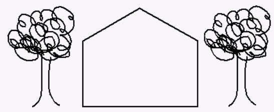


FIGURE 6

Task cards

House Puzzle

1. On a piece of paper, make a house like this one with two pieces.
2. Trace around the house you made to form a shape.
3. Make the shape with two other pieces.
4. Make the shape with three pieces. Can you find two ways to do it?
5. Can it be made with four pieces?



Tall-House Puzzle

1. On a piece of paper, make a tall house with piece 2 as the roof and one other piece.
2. Trace around the tall house you made.
3. Make the shape with pieces 5 and 7.
4. Can it be made with three pieces?

Make a Puzzle

1. Use any two, three, or four pieces. Make a shape. Trace around it on a large index card. Color it.
2. Can you make this shape with other pieces?
3. Write your name and a title for your shape on the index card.

FIGURE 5

This shape can be made in several different ways.

children might have while doing them.

Some students use strategies to solve these puzzles. For example, in part 4 of both house puzzles, children who know that rectangle piece 3 can be made from pieces 5 and 6 may use this relationship to figure out a solution by putting piece 1 or 2 on top and pieces 5 and 6 in the rectangular space on the bottom. It is important for children to share their approaches with classmates, perhaps by using an overhead projector to "show and tell." Teachers should also encourage problem posing. Children enjoy creating puzzles for others to solve. Puzzles can be presented as cutout shapes or can be drawn on cards and set out in a math center. Students can label puzzles with their names—for example, Big House Puzzle by Dina—which builds ownership of their creations.

Enlargements of pieces can be made; for example, pieces 2 and 4 make an enlargement of piece 2. Try this enlargement, and then make it with two other pieces, then with three. The enlargement has sides twice as long as piece 2, which we can readily see by making it on the triangle grid (see **fig. 7**). Using pieces 2, 4, 5, and 7, make an enlargement with sides three times as long as piece 2. Find four other pieces that work. Challenge: Make an enlargement with all seven pieces. Comparing the sides and angles of these triangles with those of piece 2, we see that the sides get progressively larger while the angles remain the same.

Exploring Geometry Shapes and Angles

Children soon notice that the sides of piece 2 have the same length, and likewise for the sides of each enlargement. So at this point, we can give a name for these figures—equilateral triangle—and ask students why the name is appropriate, that is, it has equal sides.

With this beginning, we can appreciate the advantages that this approach has for teaching geometry. First, children engage in activities that they think of as play and hence enjoy. They have puzzles to solve, and they learn things without the intention to learn. At appropriate times, teachers can introduce the names of pieces. After some time, children will use the names themselves and learn that the name remains the same no matter how the piece is placed. They also start to notice features of shapes. For example, piece 2 has equal sides; its corners are the same—are equal angles; and it looks the same when it is flipped—exhibits line symmetry—or turned—exhibits rotational symmetry. Children can learn about other pieces in a similar manner.

Next, the name *rectangle* is given for piece 3. Children are told that all three shapes in **figure 8** are rectangles, too, and asked to make them. Have children make the "tall" rectangle with pieces 1, 5, 6, and 7 and the rectangle in a "crooked" position with pieces 1 and 7 and the flip sides of pieces 5 and 6.

Can other rectangles be made? Of course, the largest is the large rectangle in **figure 1.** It is a challenge for students to reconstruct it without seeing the completed design. Children can arrange the pieces in several ways, and they enjoy finding new ways. By making various rectangles, children will—after some time—discover that all rectangles are not enlargements of one another, as was the case for equilateral triangles. Also, in contrast with equilateral triangles, the rectangle is a common everyday shape, and children should be asked to find and share examples of this shape from their home and school environments. After studying rectangles, children can investigate pieces 5 and 6, which form piece 3. These shapes are right triangles, or "rectangled triangles" as we call them in the Netherlands. Children can be asked to make other right triangles—for example, try 1, 2, 5, 6; or 3, 5, 6—and check whether they are all enlargements of piece 5.

Children can also play games that draw their attention to shapes and their parts. They could play "feel and find the shape," in which they hold a piece without seeing it and try to find the one that matches. Asking "How did you know?" encourages descriptive communication about the pieces, such as, "It has four sides and a pointy corner" for piece 7.

Fitting pieces into puzzles helps children become aware of the features of the sides and angles of the pieces. Some pieces have square corners, others have "sharp pointy" corners. Some have two equal

FIGURE 7

Enlargements of piece 2

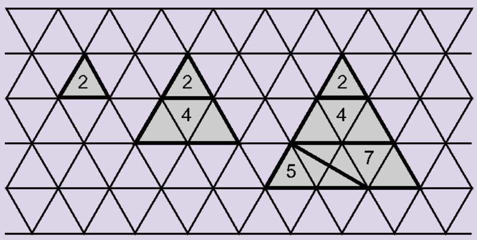
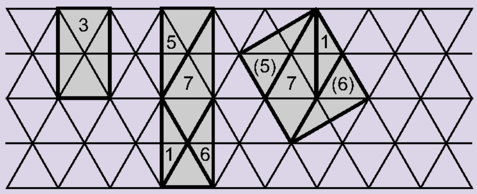


FIGURE 8

Rectangles



sides, whereas others have all equal sides or no equal sides. The language of sides and angles can now be introduced, but, of course, not with a formal definition. Students can compare triangle pieces and show how they are alike—for example, three sides, three angles—and different—for example, all sides equal, two sides equal, no sides equal, three angles the same. Piece 1 has two equal angles. What other piece has this property? Placing angles on top of each other to test whether they are equal helps children understand that the size of the angle is not dependent on the lengths of its sides.

Angles of the mosaic pieces come in five sizes. Asking children to compare angles of pieces with a square corner, or right angle, leads to informal work with acute angles—those smaller than a right angle—and with obtuse angles—those larger than a right angle. Building on the language that children invent for these kinds of angles, teachers can gradually introduce conventional terms. Children can find relationships between angles of pieces—for example, how the smallest angle relates to the other angles: they equal two, three, four, and five of the smallest. These activities are done without reference to angle measurement and build a foundation for later work with angles, their measurement in degrees, and angle relationships.

An interesting activity for children who know about angle measure is to figure out the measure of the angles in each of the seven pieces without using a protractor. Many ways are possible, and children should compare their approaches. Examine the pieces in **figure 1,** and find the measures of the angles of each piece. Think about the angle relationships that you used and whether you could figure out these measures in other ways by using other angle relationships.

Children who use the triangle grid to record solutions to puzzles become aware of equal angles in the grid and also of parallel lines. They can be asked to look for lines like train tracks and trace them with different-colored markers, creating designs that show three sets of parallel lines. Parallelism of lines is a feature needed for describing pieces 4 and 7—trapezoids, which have one pair of parallel sides—and also applies to the opposite sides of piece 3, a rectangle.

Other Activities with the Mosaic Puzzle

Placing pieces to fill in the space in puzzles also provides experiences with area. By direct comparison, students can show that some pieces take up more space than others—piece 7 has a greater area than piece 2—or can discover relationships, such as, piece 5 is half of piece 3. Working with shapes on the triangle grid reveals other relationships, such as, piece 4 has three times the area of piece 2, or how the area of piece 2 compares with the area of its enlargements (see **fig. 7**). A similar exploration of area could be done with piece 4 and its enlargements. These kinds of experiences with area lay a foundation for later work with square units of area and the discovery of ways to find the area of various shapes—for example, why the area of a right triangle is one-half the area of a rectangle—and how the enlargement of a shape, for instance, by doubling the lengths of its sides, affects its area.

To further develop children's descriptive thinking about the pieces, they can play "clue" games about the pieces or the shapes they made with them. Clues for piece 4 could be "four sides, four angles, two equal sides, two equal acute angles, and two parallel sides." Clues are revealed one at a time until the shape is identified. After each clue, children tell which pieces work or do not work and explain why. They could also play "guess the piece," in which they ask the teacher yes-no questions about the mystery shape. The teacher can list questions on the chalkboard and have children discuss whether all are needed to identify the shape. Children may point out that some properties imply others, such as "three sides" means that the shape has "three angles." These kinds of games give practice with properties that children have learned so far and strengthen children's use of descriptive language as a tool for reasoning about shapes and their properties. They also give teachers a window to children's developing levels of thinking, here between the descriptive level and the next level, where properties are logically ordered.

Having played with this special mosaic in these activities, we sense that many other questions to pose and topics to explore are possible. Furthermore, grids and mosaics based on other types of shapes can be used, such as one based on squares, leading in a natural way to area and to coordinate geometry, which connects shape and number.

Reflections on the Activities and Looking Ahead

Activities with mosaics and others using paper folding, drawing, and pattern blocks can enrich children's store of visual structures. They also develop a knowledge of shapes and their properties. To promote the transition from one level to the next, instruction should follow a five-phase sequence of activities.

Instruction should begin with an **inquiry phase** in which materials lead children to explore and discover certain structures. In the second phase,

direct orientation, tasks are presented in such a way that the characteristic structures appear gradually to the children, for example, through puzzles that reveal symmetry of pieces or through such games as "feel and find the shape." In the third phase, **explicitation,** the teacher introduces terminology and encourages children to use it in their conversations and written work about geometry. In a fourth phase, **free orientation,** the teacher presents tasks that can be completed in different ways and enables children to become more proficient with what they already know, for example, through explorations of making different shapes with various pieces or through playing clue games. In the fifth and final phase, **integration,** children are given opportunities to pull together what they have learned, perhaps by creating their own clue activities. Throughout these phases the teacher has various roles: planning tasks, directing children's attention to geometric qualities of shapes, introducing terminology and engaging children in discussions using these terms, and encouraging explanations and problem-solving approaches that make use of children's descriptive thinking about shapes. Cycling through these five phases with materials like the mosaic puzzle enables children to build a rich background in visual and descriptive thinking that involves various shapes and their properties.

Remember, geometry begins with play. Keep materials like the seven-piece mosaic handy. Play with them yourself. Reflect on what geometry topics they embody and how to sequence activities that develop children's levels of thinking about the topics. Then engage your students in play, activities, and games that offer an apprenticeship in geometric thinking. Children whose geometric thinking you nurture carefully will be better able to successfully study the kind of mathematics that Euclid created.

Watch for "Investigations: Are You Puzzled?" by Rosamond Welchman in the March 1999 issue for a related puzzle activity.

References

Fuys, David, Dorothy Geddes, and Rosamond Tischler. *The van Hiele Model of Thinking in Geometry among Adolescents, Journal for Research in Mathematics Education* Monograph Series, no. 3. Reston, Va.: National Council of Teachers of Mathematics, 1988.

van Hiele, Pierre M. *Structure and Insight.* Orlando, Fla.: Academic Press, 1986.

———. *Structuur* (Structure). Zutphen, Netherlands: Thieme, 1997. ▲

BY WAY OF INTRODUCTION

(Continued from page 307)

would like to revise slightly Euclid's comment: "There is no royal road to teaching geometry"; it takes hard work by dedicated teachers to give students quality instruction. With this thought in mind, our hope is that you will find articles in this 1999 focus issue that will be useful in your journey along the road to accomplishing this worthy task.

Charles Geer
For the Editorial Panel

The Editorial Panel's commitment to features on geometry and geometric thinking goes beyond this focus issue. Be on the watch for future articles about this increasingly important mathematics topic, and please consider sharing your own ideas with the Panel. In particular, watch for the following articles that will be published in future issues of Teaching Children Mathematics*: "The Importance of Spatial Structuring in Geometric Reasoning," by Michael T. Battista; "Geometry and Op Art," by Evelyn J. Brewer; "Why Are Some Solids Perfect? Conjectures and Experiments by Third Graders," by Richard Lehrer and Carmen L. Curtis; and "Getting Students Actively Involved in Geometry," by Stuart P. Robertson.* ▲

Sorting objects using Geostacks

Free Play Activity

Children stack five shapes of different colour onto rods. These could be stacked according to shape or colour. Teachers may ask learners the names of the shapes or the colours of the shapes.

Teachers could ask the children why they chose to stack the shapes as they did.

Focused Play Activity

Cards are provided with specific activities for learners to follow. Learners have to stack the shapes according to the directions on the card.

1. According to colour
2. According to shape *and* colour
3. Stack cards according to colour in the order shown on card
4. Stack shapes according to shape in the order shown on card.
5. Stack shapes according to shape *and* colour in the order shown on card.
6. Understanding quantity – Stack shapes according to the number of dots on the card.
7. Stack shapes according to the number of dots on the card *and* according to colour given.
8. Stack shapes according to the number of dots on the card *and* according to specific shape given.
9. Introducing numerals – Stack shapes according to numerals given.

Should you wish to purchase a Geostacks set, please refer to the Resource list.

Bead Patterns

Free Play Activity

3-D objects of different size, shape and colour are thread on a lace. During free play, learners can make a “pretty necklace”.

Focused Play Activity

Cards are provided with specific activities for learners to follow during focused play.

1. Learners have to thread the beads and continue a pattern according to the directions on the card.
   a. According to shape.
   b. According to shape and size.
   c. According to shape, size and colour

   Teachers can facilitate learning by asking, “Which are bigger/smaller?” and “Which of these shapes roll?”

2. Objects are arranged to form patterns given on cards (without threading). These patterns include translations, rotations and reflections. Children must first make the patterns using the beads. The cards include questions for the teacher to facilitate discussion. For example:
   a. Name two shapes created in the illustration.
   b. What fraction does blue represent in the illustration?
   c. How many small cubes will be used to equal the length of 4 large cubes?
   d. How many triangular prisms were used to create this hexagon?
   e. Name two attributes shared by the first two shapes in the illustration?

Should you wish to purchase beads and sets of cards, please refer to the Resource list.

Attribute blocks

Free Play Activity

Attribute block consist of shapes (triangles, squares, rectangles, circles and hexagons) in different colours, size and thickness. During free play, learners can build boats, people, houses or any imaginative pattern.

Focused Play Activity

During focused play, learners copy patterns on cards. The cards focus attention on type of shape, colour, size and thickness. They also provide opportunity for learners to notice position of objects, translations, rotations and reflections, and to continue patterns. The cards include questions that a teacher can ask to further facilitate learning. For example:

1. How many yellow shapes?
2. How many rectangles?
3. How many thick shapes?
4. What are the similarities/differences between two shapes in an arrangement?
5. Compare the heights of the stacks. Are they equal?

Should you wish to purchase a set of attribute blocks and cards, please refer to the Resource list.

Using pegboards

Free play Activity

During free play, learners can make pictures by placing coloured pegs in holes on a peg board.

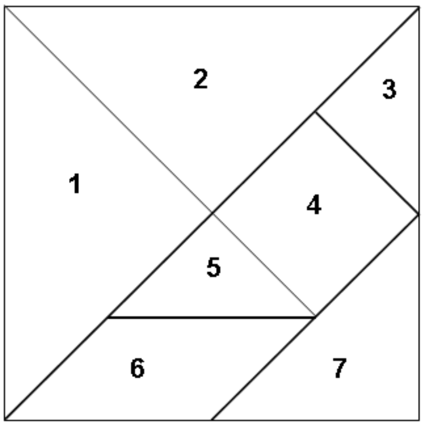
Focused play Activity

Even at a young age, learners can start focused play by copying a picture, or copying and completing a pattern on activity cards. As the activities get more complicated, learners will need to use rotation and reflection to complete patterns.

Should you wish to purchase pegboards and sets of cards, please refer to the Resource list.

Properties of 2-D shapes using Tangrams

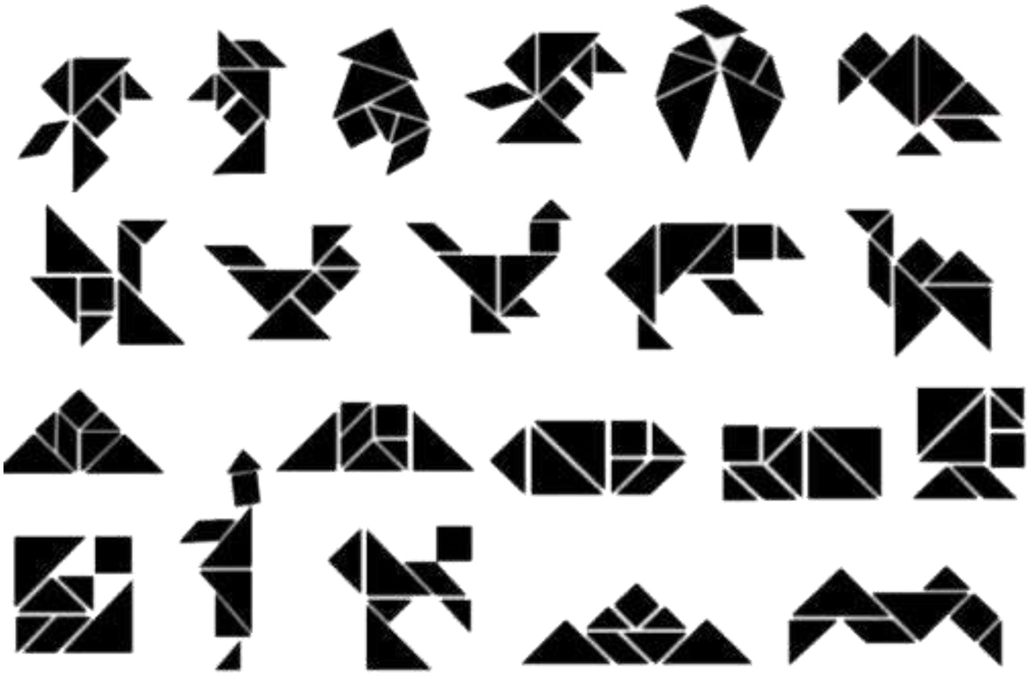
The Tangram is an ancient game which originated in China. A set of 7 basic shapes is cut from a single square – 5 triangles of different sizes, 1 square and 1 parallelogram.



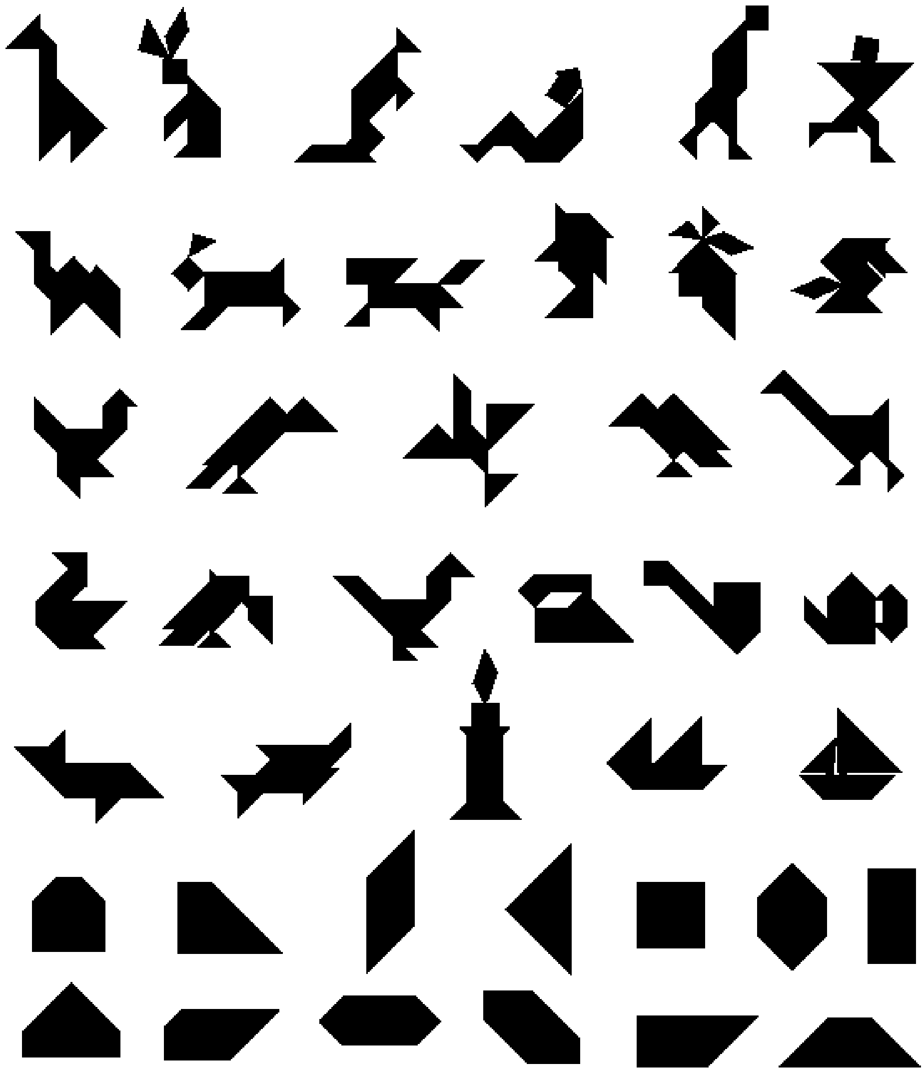
Free Play Activity

Children can put the pieces together to make any imaginative figures or pictures.

Children can copy pictures like these:



For a more challenging option, children can use the tangram pieces to build pictures like these (that do not show individual pieces):



More examples are provided on Tangram Activity 1.

Focused Play Activity

Examples of how tangrams are used for focused play are provided in Tangram activity 2 to 5 on the following pages.

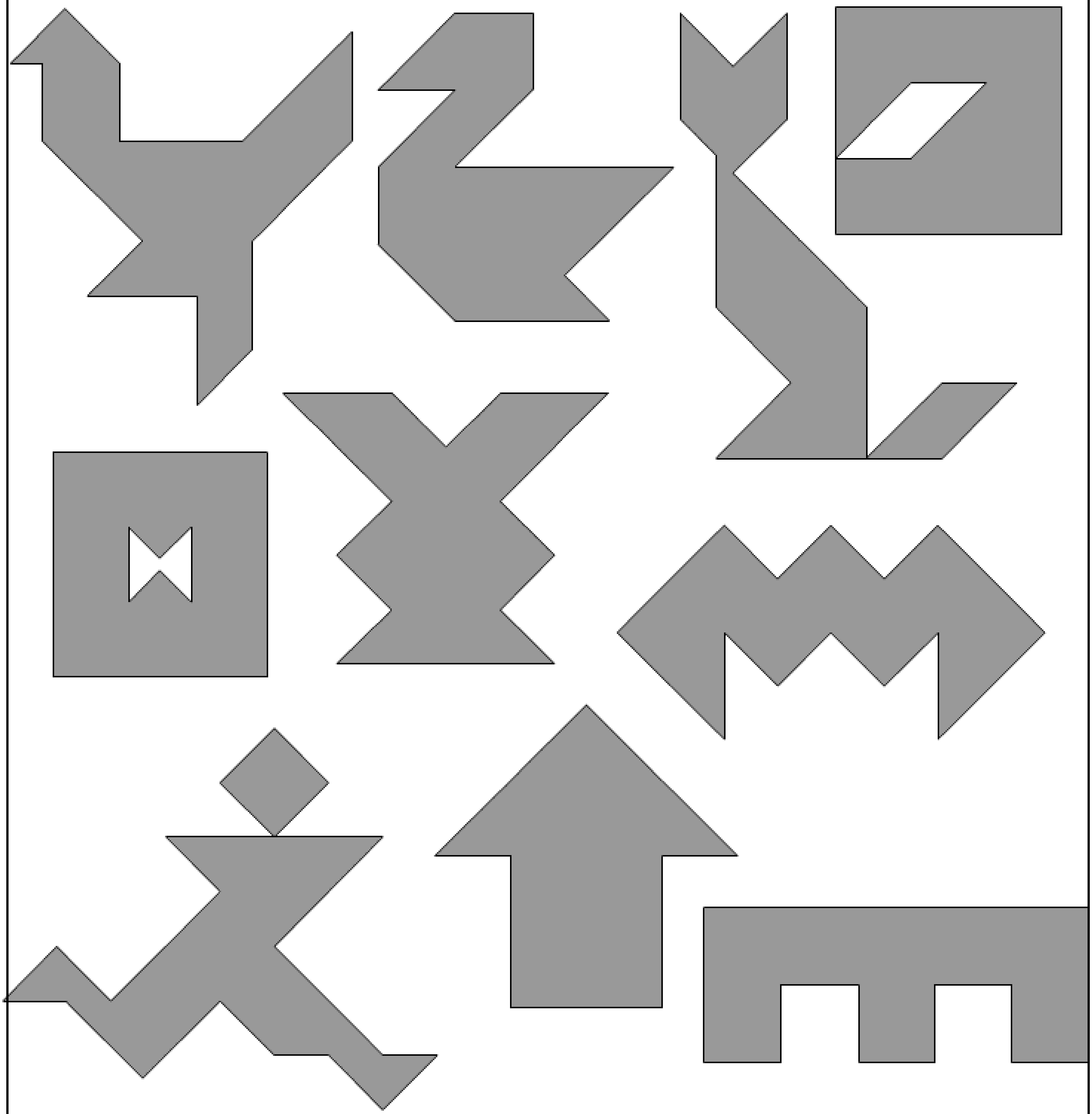
Tangram puzzles and activity cards can be purchased. Please refer to the Resource list.

Alternatively, tangram puzzles can be made. See the template provided. Print the template on cardboard, cut out the pieces and laminate.

Activity cards or sheets are needed for focused play. There are many free resources available for download on the web. Also see Tangram Activities 1 to 5 on the following pages

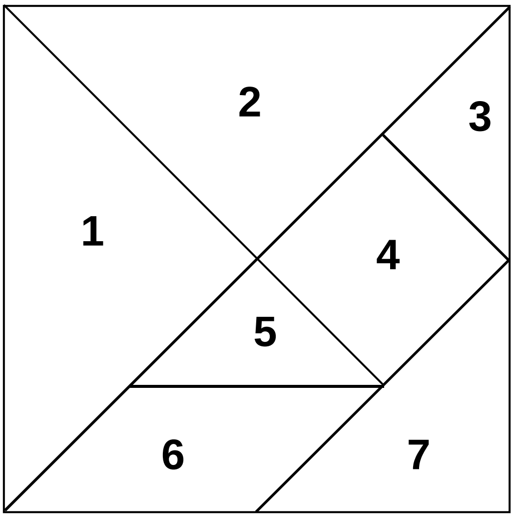
Tangram Activity 1*

Challenge: Make each of these shapes with the pieces of one Tangram puzzle.



* Adapted from material developed by MALATI. Materials developed by MALATI are in the public domain and may be freely used and adapted, with acknowledgement to MALATI and the Open Society Foundation for South Africa.

Tangram Activity 2: Making New Figures*



1. Use figures 3 and 5 to make a square the same size as figure 4.
   How do you know that the figure you have made is a square?

2. Use figures 3 and 5 to make a triangle the same size as figure 7.

3. Use figures 3, 5 and 7 to make a square.
   How do you know that the figure you have made is a square?

4. Use figures 3, 4 and 5 to make a rectangle.
   How do you know that the figure you have made is a rectangle?

5. Use figures 3, 4 and 5 to make a triangle the same size as figure 2.
   Is there only one way?

6. Now use figures 3, 5 and 6 to make a rectangle.
   How do you know that the figure you have made is a rectangle?
   Is there more than one way?

Tangram Activity 3: Making New Figures*

1. Use figures 3 and 5 to make a figure the same size and shape as figure 6. Do you know the name of this figure?

2. Use figures 3, 5 and 6 to make a figure the same shape as this:
   Is there more than one way? What is the name of the figure?

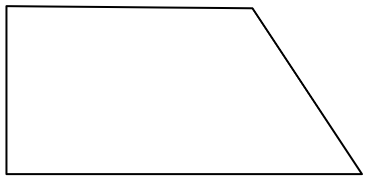
3. Use pieces 1, 3, 4 and 5 to make the following figures:

   (a) Square (b) Parallelogram (c) Isosceles triangle (d) Trapezium

4. Is it possible to make a rhombus from pieces 1, 3, 4, and 5?

5. Use pieces 1 and 7 to make a figure the same shape as this:

   What is the name of the figure?



6. Use pieces 1, 2, 3, 5 and 6 to make a rectangle.

7. Use **smaller** figures to make each of the following figures. In each case explain how you know you have made required figure.

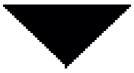
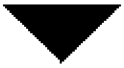
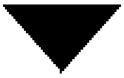
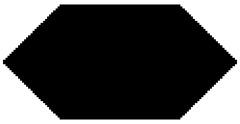
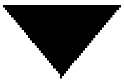
   (a) figure 1

   (b) figure 7

8. Yusuf says it is not possible to make the square made with figures 1 and 2 **without using figure 1 or 2**. Do you agree with Yusuf?

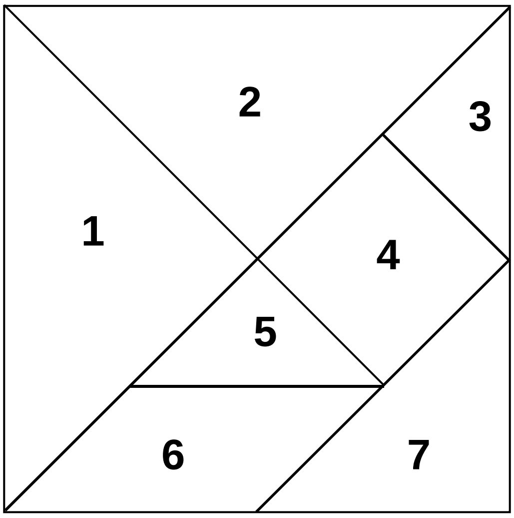
9. Now use your tangram pieces to make as many different geometrical figure as you can. Try to give each figure a name.

Tangram Activity 4: Geometrical figures and fractions*

1. How many  are in  ?
2. How many  are in  ?
3. How many  are in  ?
4. How many  are in  ?
5. If  = 1, then  =
6. If  = 1, then  =
7. If  = 1, then  =
8. If  = 1, then  =

* Adapted from material developed by MALATI. Materials developed by MALATI are in the public domain and may be freely used and adapted, with acknowledgement to MALATI and the Open Society Foundation for South Africa.

Tangram Activity 5: Geometrical figures and fractions*



1. If Triangle 1 is $\frac{1}{4}$ of the whole tangram, what fraction of the whole tangram is Triangle 2?

2. What fraction of the whole tangram is Triangle 7? Why do you say so?

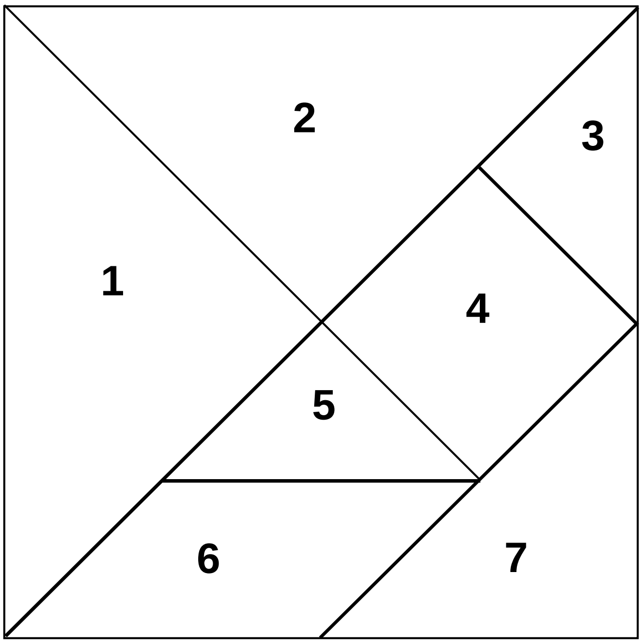
3. Why is Triangle 3 called $\frac{1}{16}$ of the whole tangram?

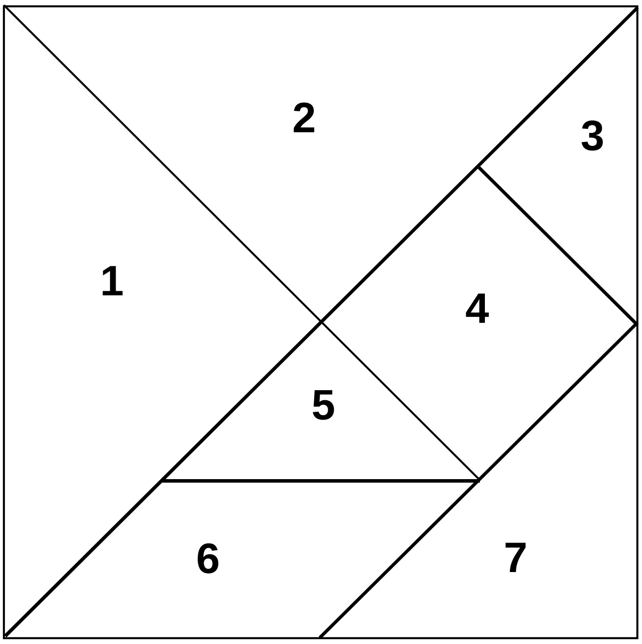
4. What fraction of the whole tangram is the square (shape 4)?

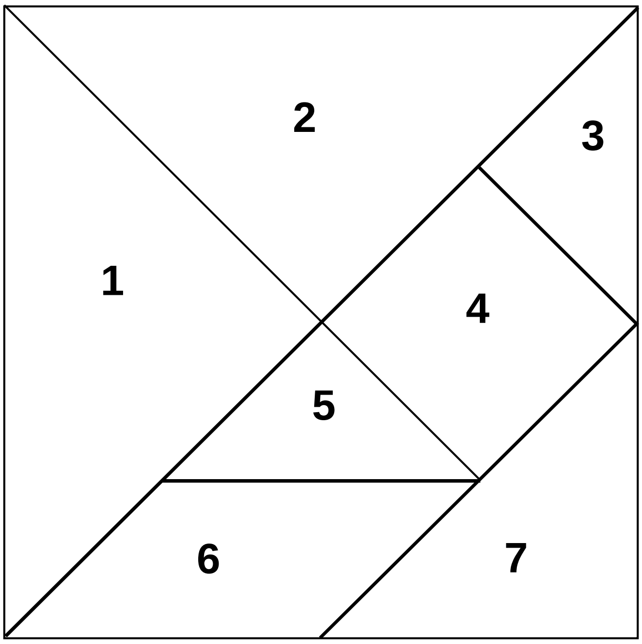
5. What fraction is the parallelogram of the whole tangram?

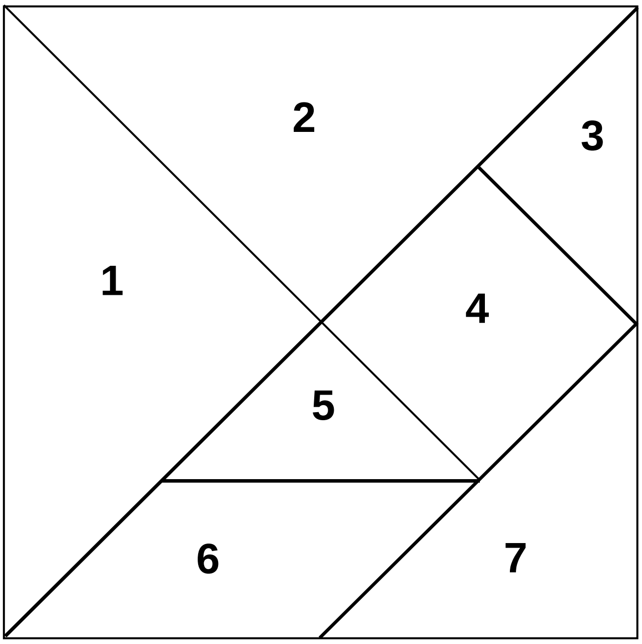
6. Can you write a fraction name for each tangram piece and show that, if you add all the fractions you will get 1?

Tangram template









* Adapted from material developed by MALATI. Materials developed by MALATI are in the public domain and may be freely used and adapted, with acknowledgement to MALATI and the Open Society Foundation for South Africa.

Viewing objects: Building structures using connecting cubes

Learners use unfix cubes to build shapes that are represented as a 2-D drawing.

Free Play Activity:

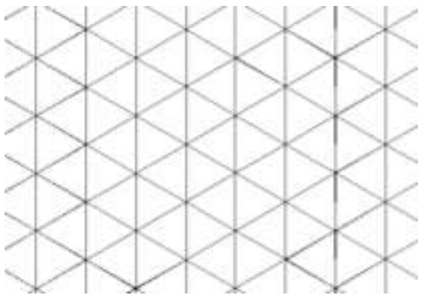
Learners can use the blocks to create imaginative structures.

Focused play Activity

Learners are given cards, like the ones on the following page. They build the structure shown on the card and ask a friend or teacher to check before continuing to the next picture. The cards get progressively more difficult.

Extension Activity:

Learners can build any structure and then represent this structure on isometric paper.



1. *Should you wish to purchase connecting cubes, please refer to the Resource list.*
2. *Activity cards are provided on the following page. These can be printed on cardboard and laminated.*
3. *Isometric paper is available from most specialist stationery shops. You can also download and print it free from the following websites:*

    http://www.isometricpaper.co.uk/

    http://www.printablepaper.net/category/dot

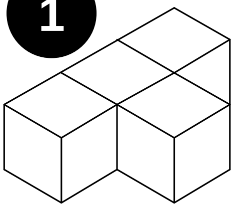
    http://www.mathsphere.co.uk/resources/MathSphereFreeGraphPaper.htm

Activity cards for building structures using connecting cubes

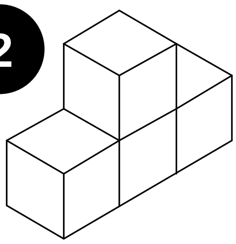
Use unifix cubes to make the following:

Card A

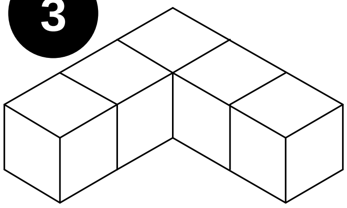
1



2

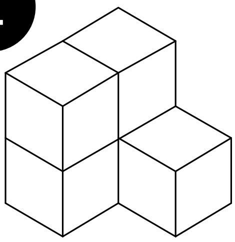


3

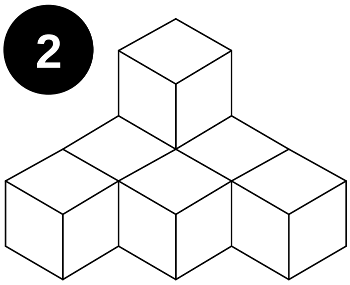


Card B

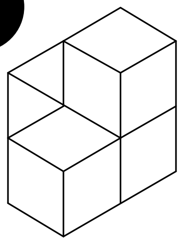
1



2

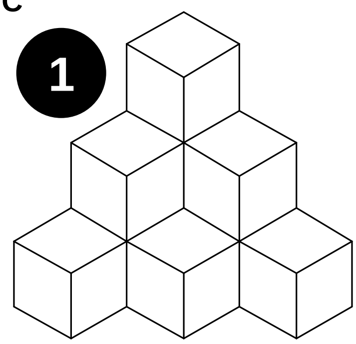


3

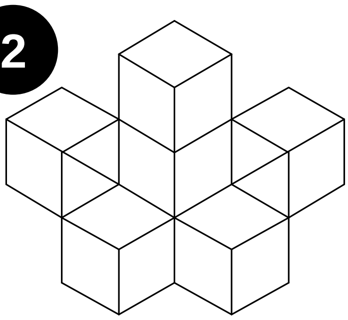


Card C

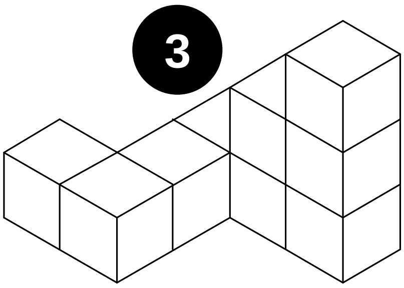
1



2

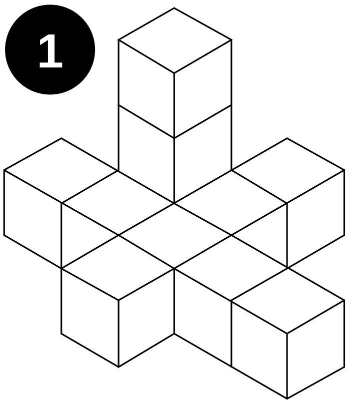


3

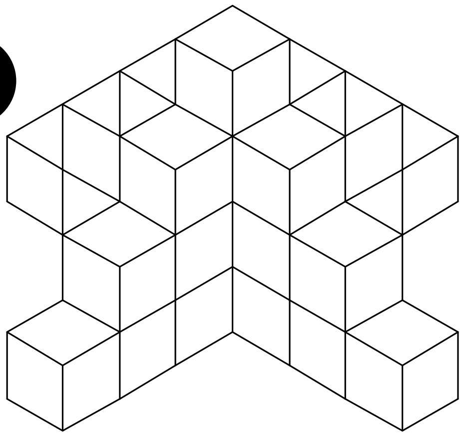


Card D

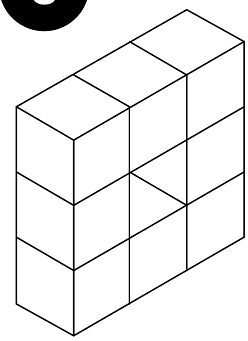
1



2



3



Viewing objects: Building structures using blocks

Free Play Activity:

Children use cardboard blocks to build shapes that are represented in 2-D drawings.

Focused Play Activity:

During focused play, learners are given cards with the drawings like the ones on the following page. They build the structure shown on the card and ask a friend or teacher to check before continuing to the next picture. The cards get progressively more difficult.

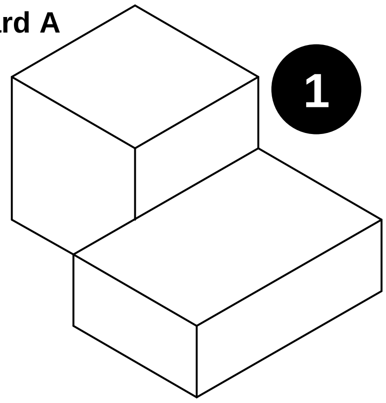
1. *An example of activity cards can be found on the following page. These can be copied onto cardboard and laminated.*
2. *Nets of blocks can be found on the following pages. These can either be constructed by the teacher or the learners. Copy the nets onto cardboard. Do not change the size of the templates supplied. Cut them out. Score along the edges and paste together using Bostick glue.*
   These blocks and cards will be available for sale in the near future. Please refer to the Resource list.

Viewing objects: Building structures using blocks

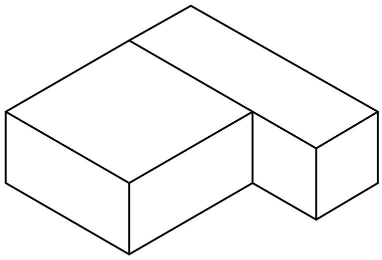
Use the blocks that you have made to make the following:

Card A

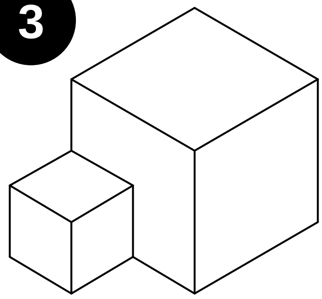
1



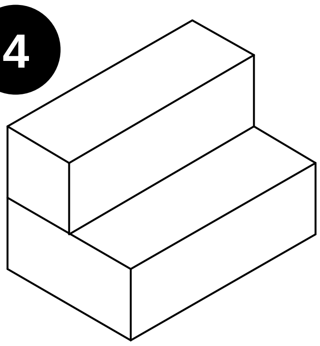
2



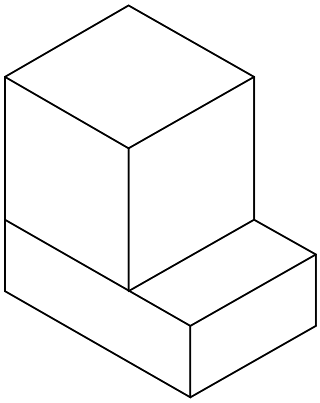
3



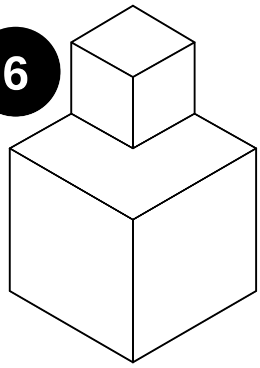
4



5

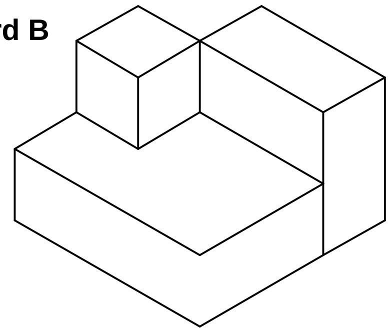


6

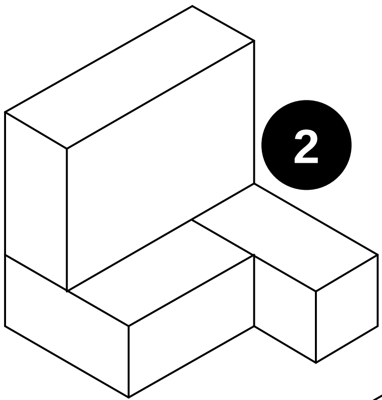


Card B

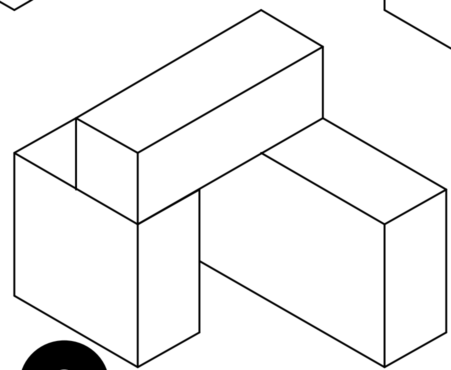
1



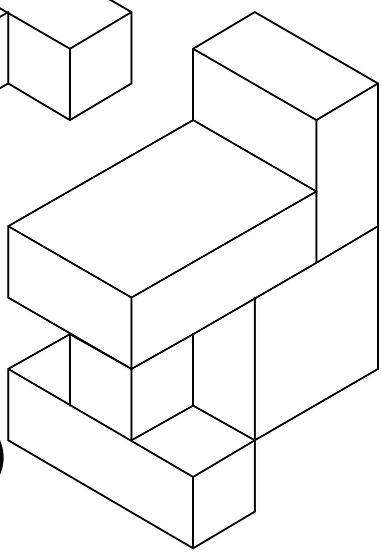
2



3

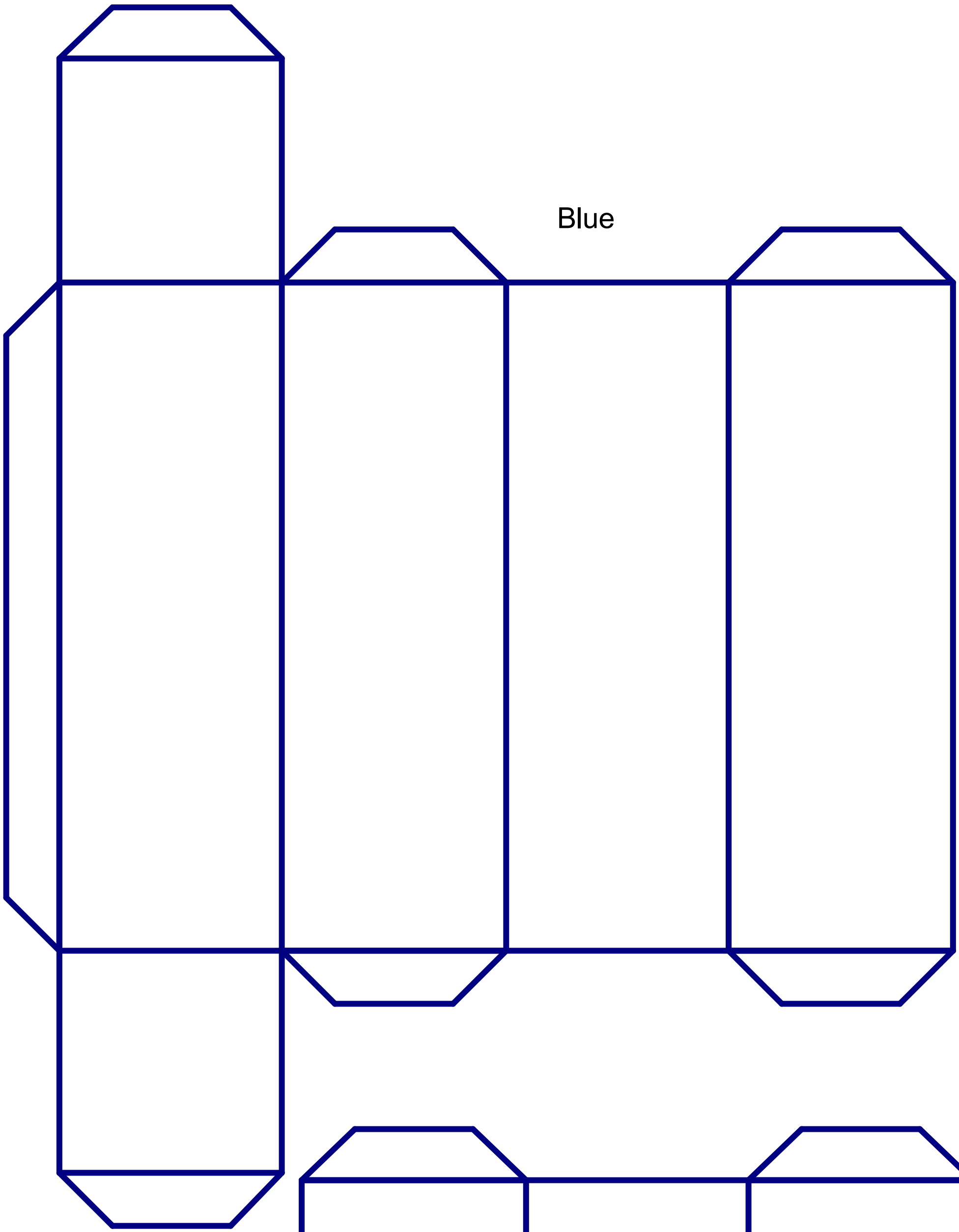


4

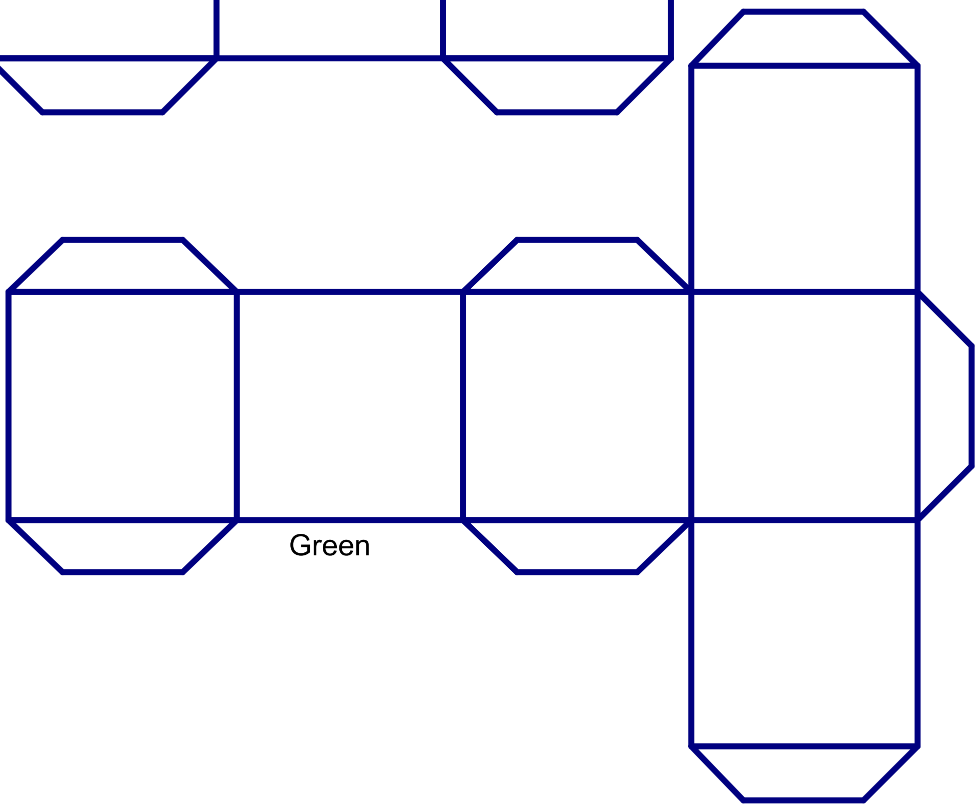


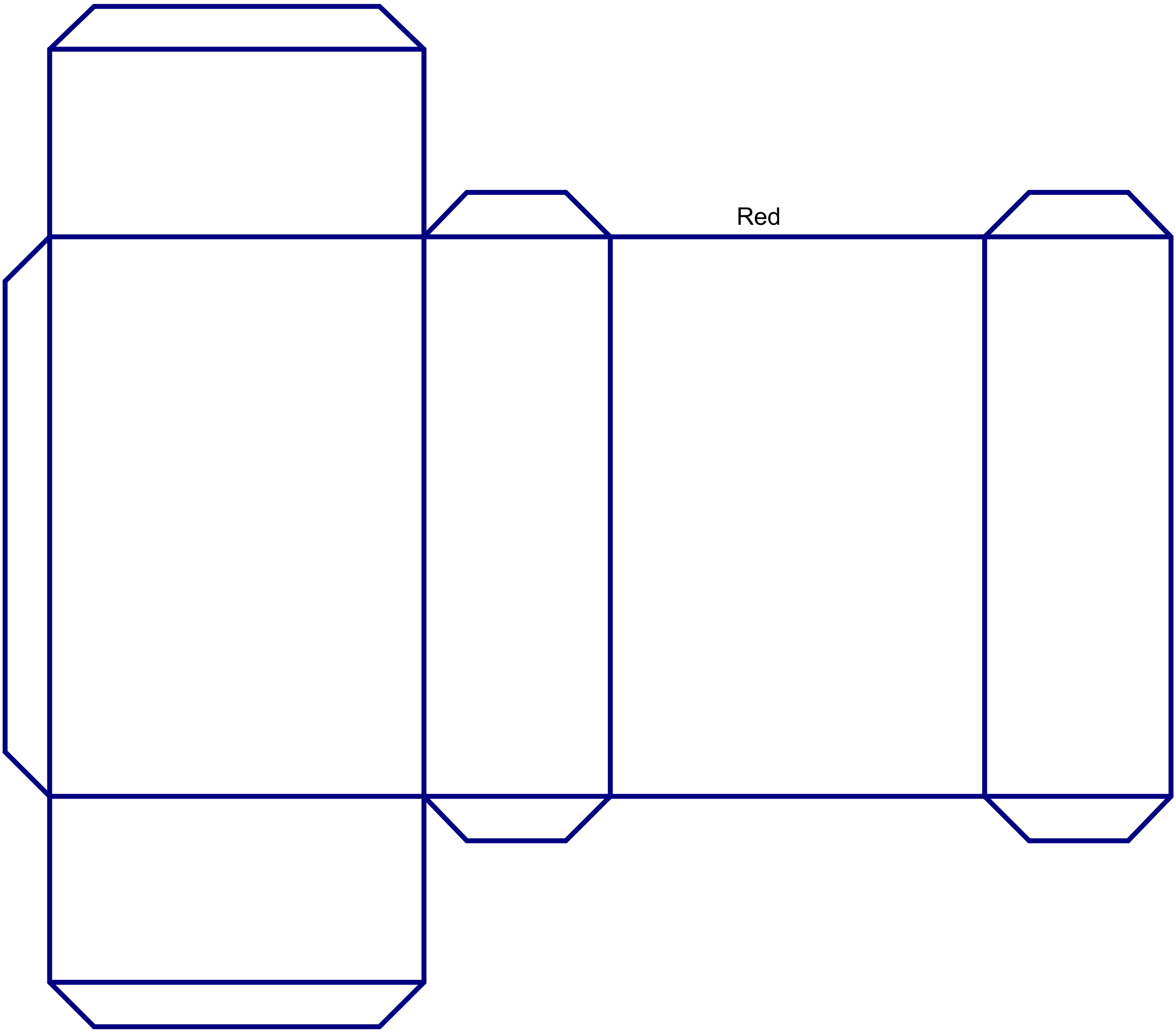
Prism Nets

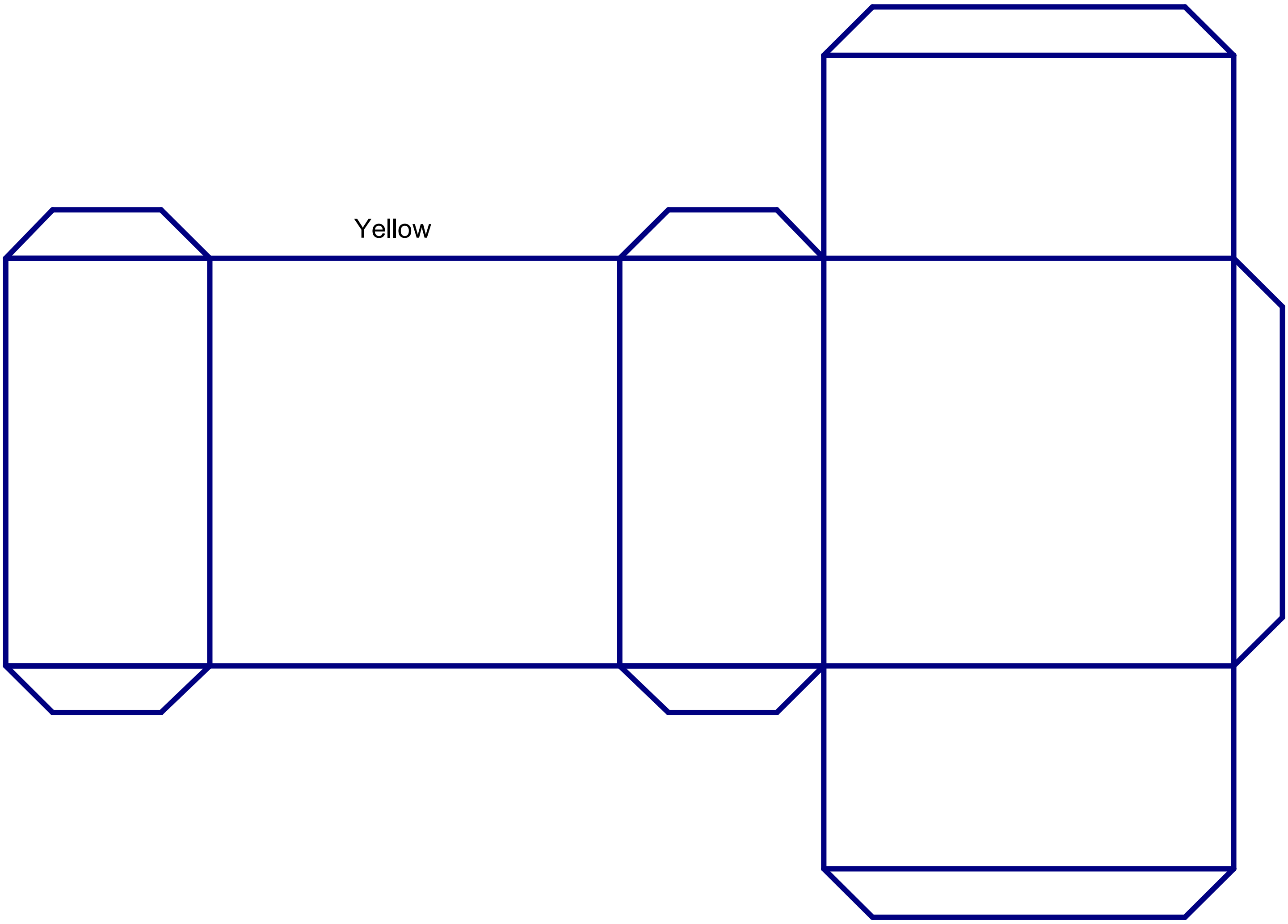
Blue

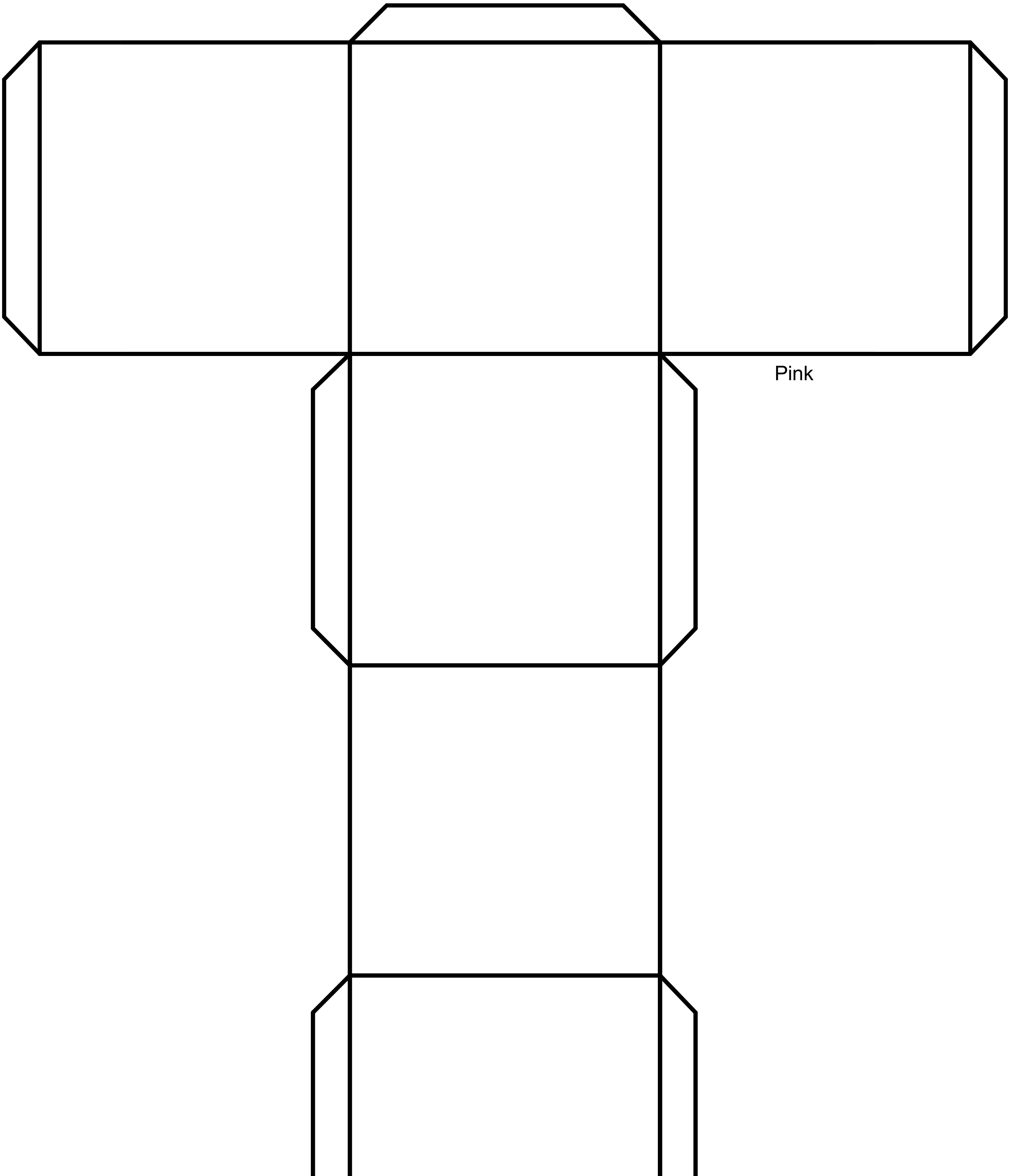


Green




Red


Yellow


Pink

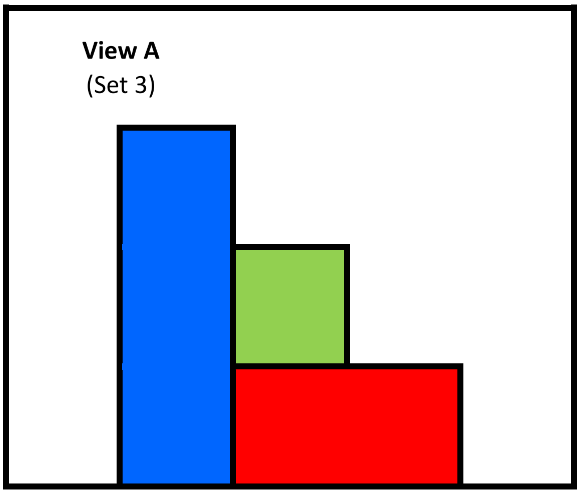
Viewing objects: Different points of view

Focused play Activity:

Work in a group of four.

Each child should sit on one side of the grid (*supplied on next page*) – either A, B, C or D. Each child should have the view card (*as supplied on the following pages*) that corresponds to their position. In other words, the child sitting at A should have a card that says “View A”.

The Viewing cards represent how the blocks are placed on the grid according to a particular view. For example, according to the card below three blocks are used – blue, red and green. The blue block has been placed upright, the red block flat and the green block on top of the red block. It also shows that there is a space on the left of the blue block and a space on the right of the red block. The card does NOT show how close or how far the blocks are placed from the viewer. Note: It is possible that a fourth block is also used that is hidden behind the three shown on this card.



Any of the four children may start the game. They choose the blocks that they think are used for the view shown on their card. They place these blocks on the grid so that the view that they see from their position is the same as that shown on their card.

The next three children take it is turns to move the blocks. They work together until all four children agree that they blocks on the grid look the same as the view represented on their card.

The cards get progressively more difficult, starting with placing only one of the prisms in the correct position on the grid (Set 1), until all five are placed on the grid (Set 7).

Extension Activity 1:

For a more challenging option, the viewing cards can be printed in black and white.

Extension Activity 2:

Children are given blank viewing cards (like those supplied). The child then places the blocks on the grid in any way. The children should then draw the corresponding viewing cards.

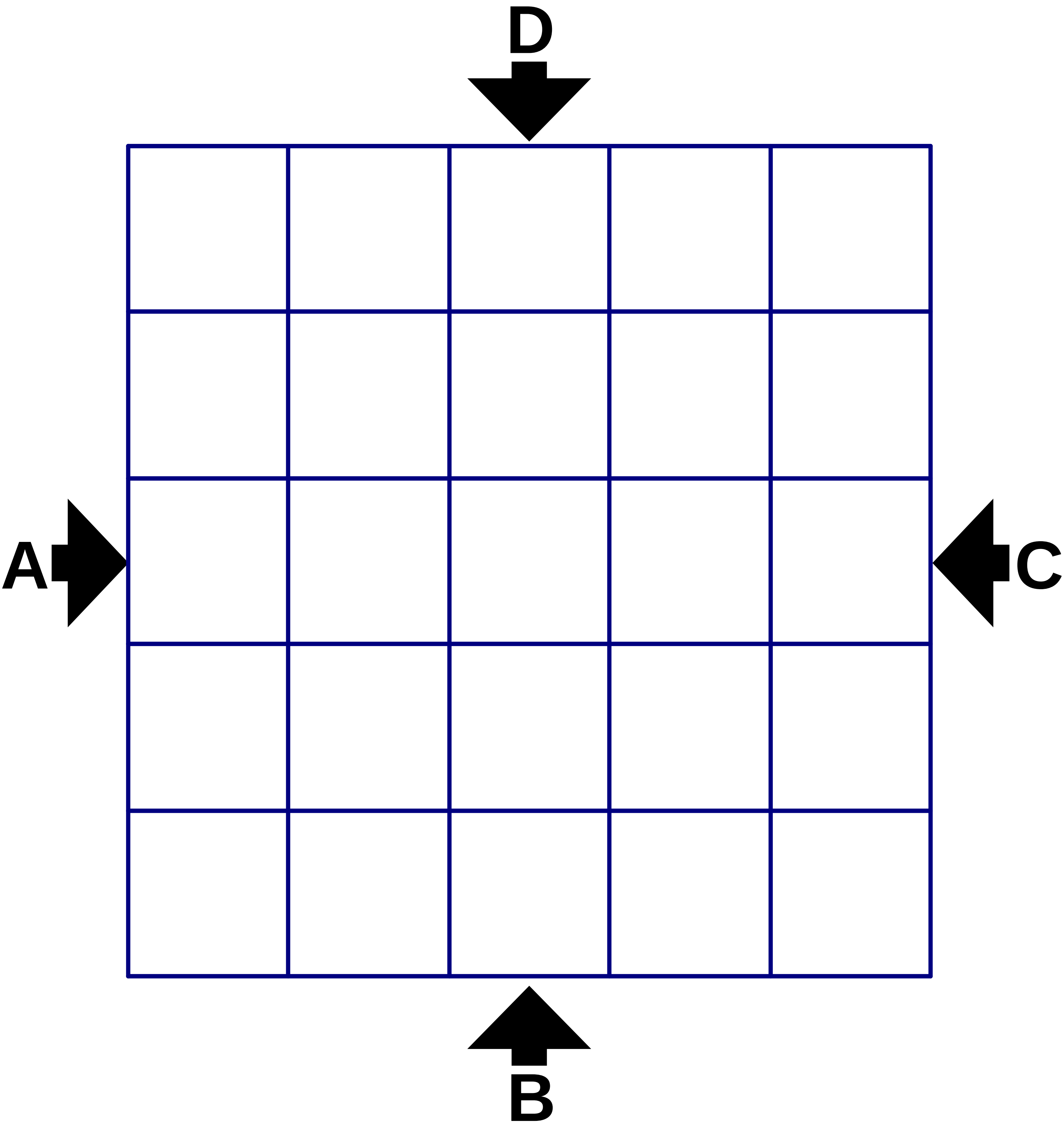
To check, ask another group to place the blocks on the grid using the view cards that the child has created.

1. *The blocks used in this activity are the same as those described in the previous activity. Blocks should be constructed using the nets supplied in the suggested colour. Do not change the size of the templates supplied.*
2. *Grid - as supplied on the following page. This should be copied on cardboard for each group and laminated.*
3. *Viewing cards - as supplied on the following pages. These can be copied onto cardboard in colour or in black-and-white (for a more challenging option) and laminated.*
4. *Set of blank viewing cards for each learner - as supplied on the following pages.*

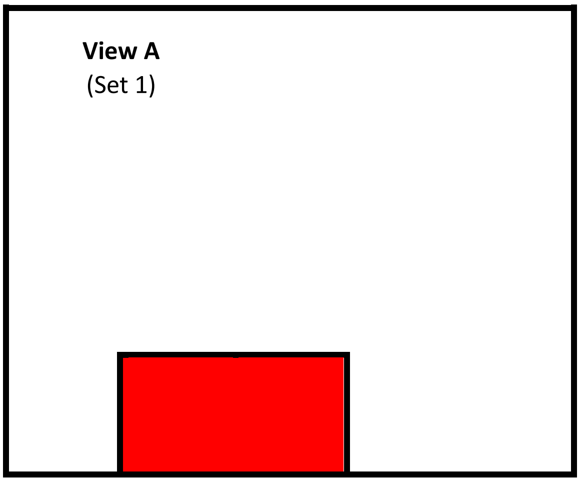
These blocks and cards will be available for sale in the near future. Please refer to the Resource list.

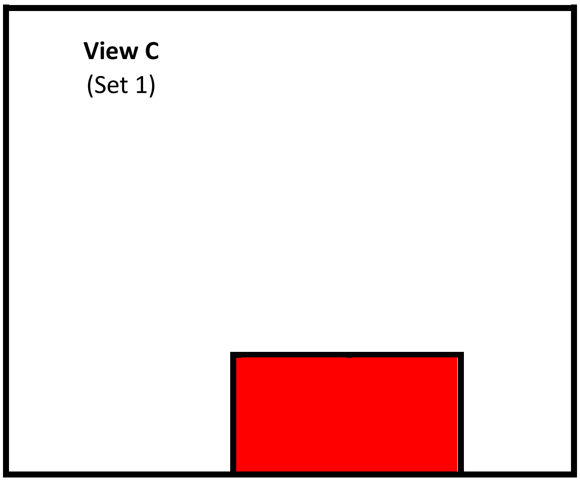
Viewing objects: Different points of view

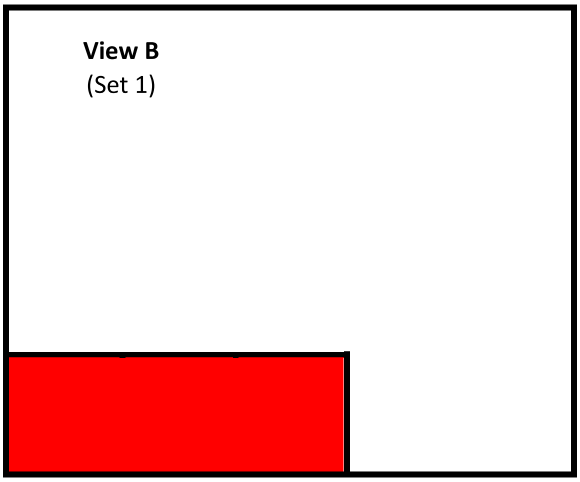
Grid

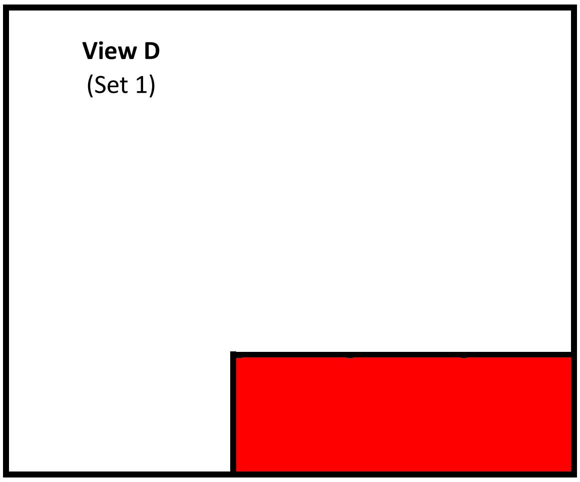


Set 1

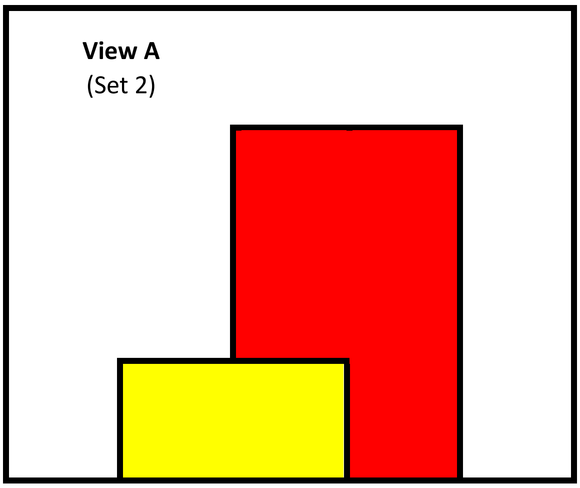

View A
(Set 1)

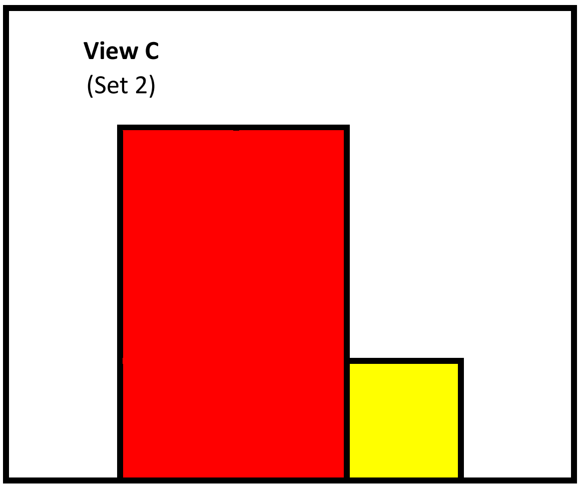

View C
(Set 1)


View B
(Set 1)

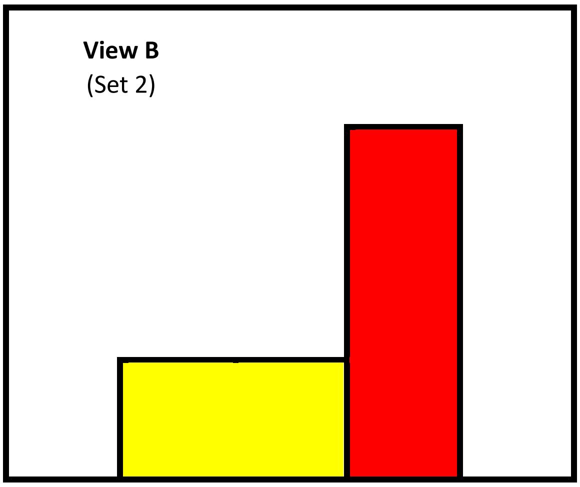

View D
(Set 1)

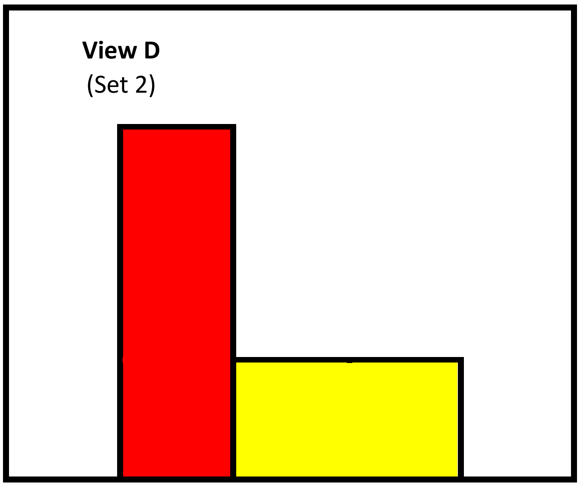
Set 2


View A
(Set 2)

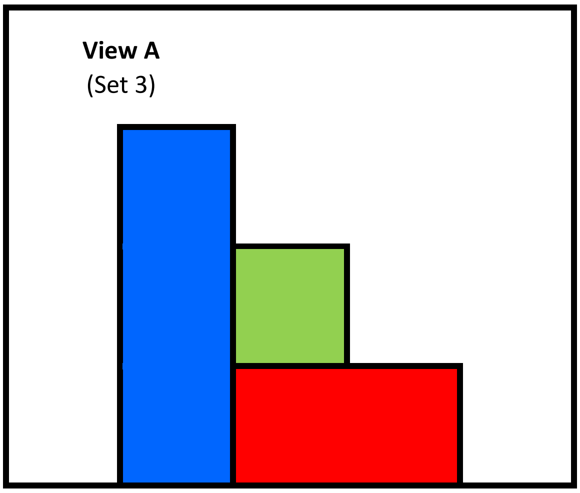

View C
(Set 2)

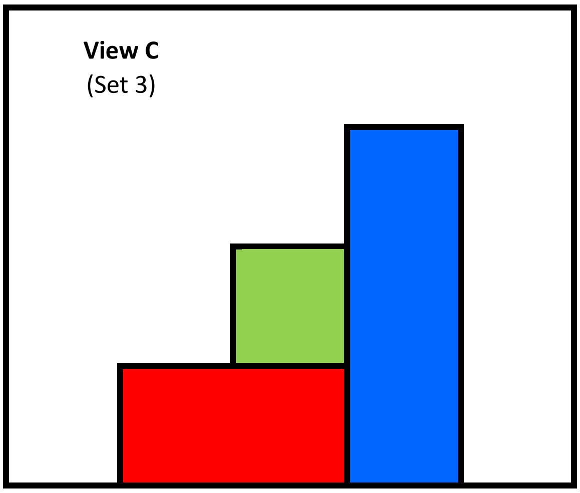
Set 2 (continued)

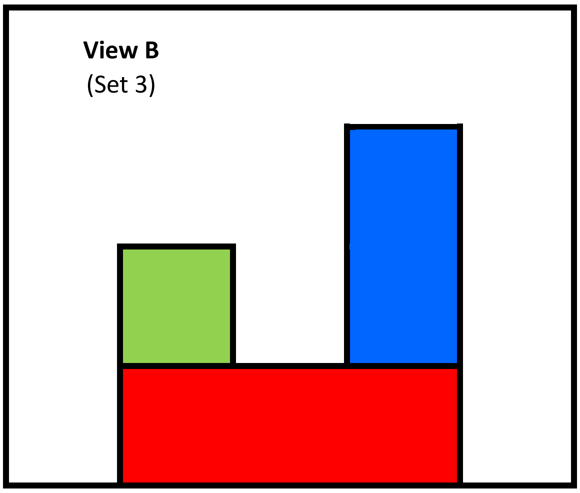


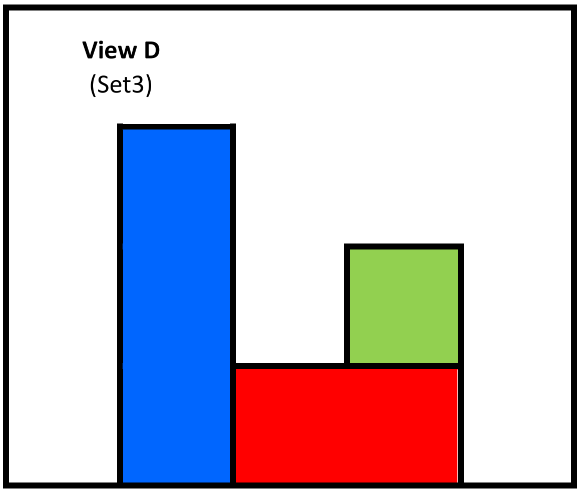


Set 3

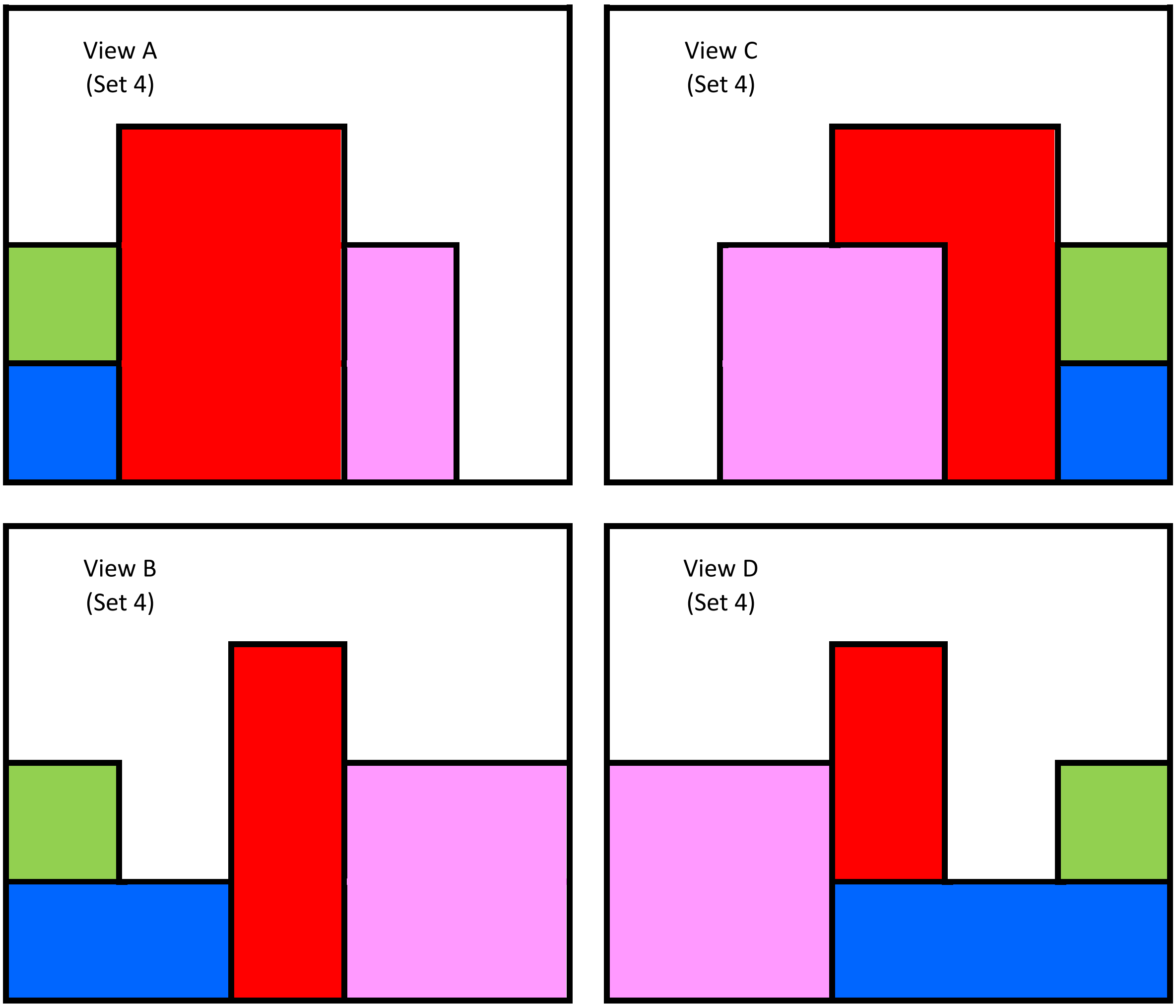




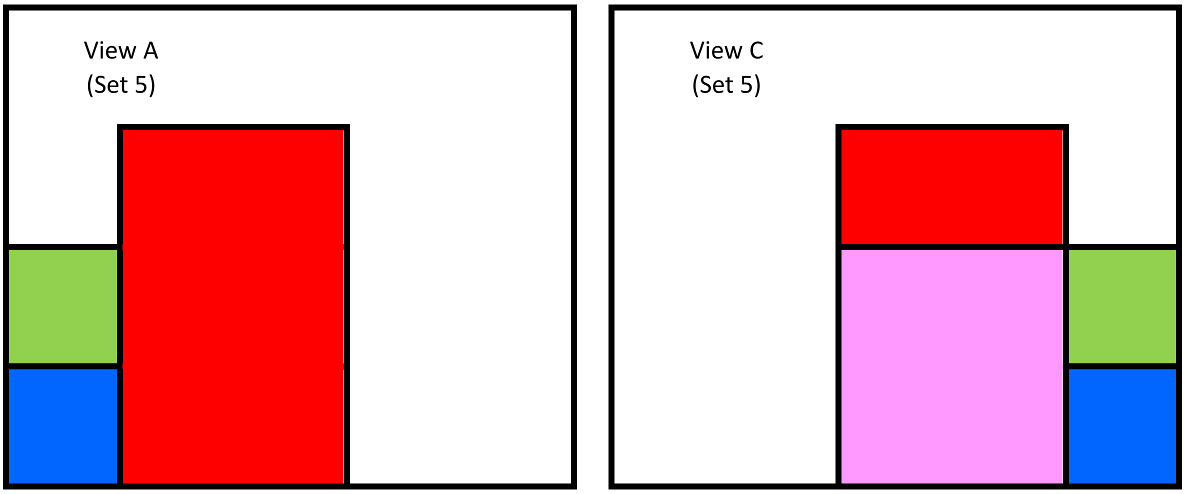




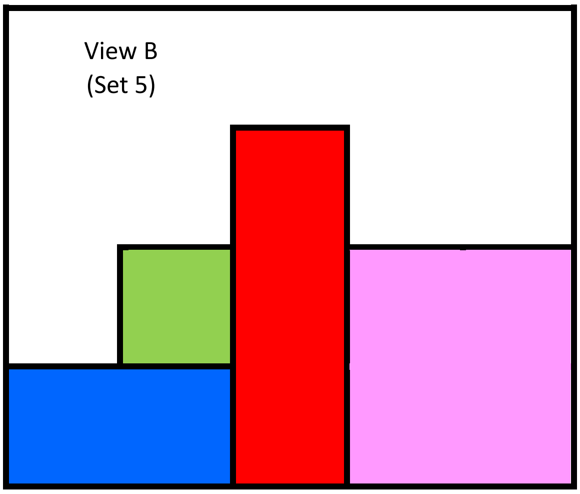
Set 4

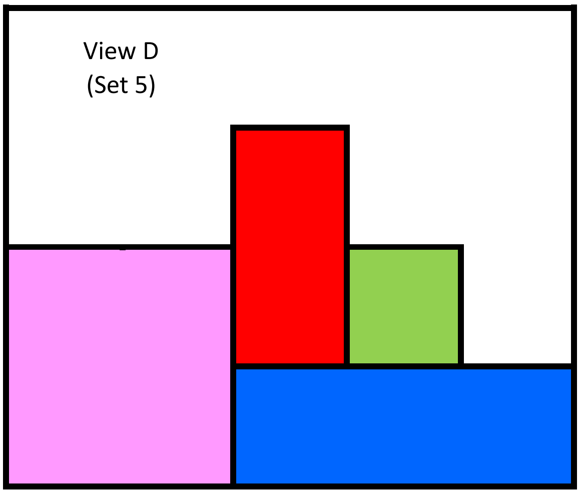

View A
(Set 4)
View C
(Set 4)
View B
(Set 4)
View D
(Set 4)

Set 5

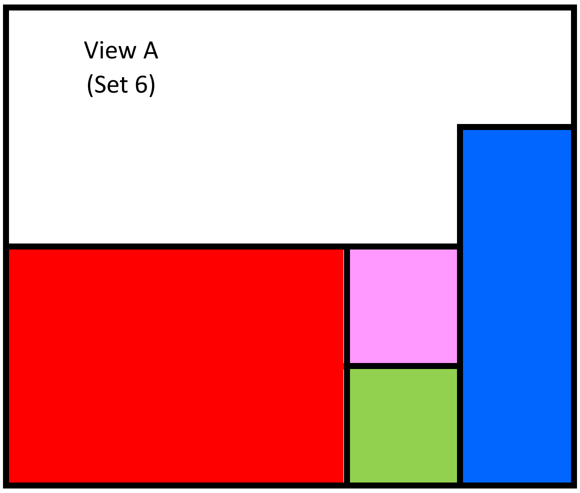

View A
(Set 5)
View C
(Set 5)

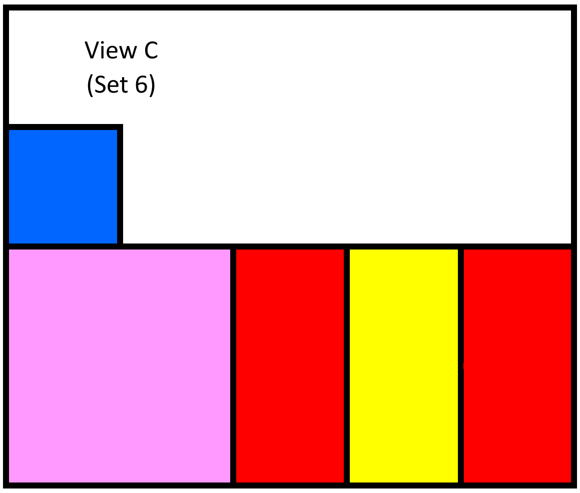
Set 5 (continued)

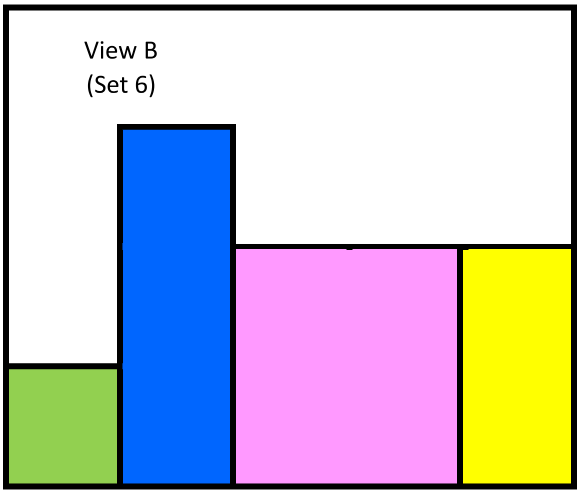


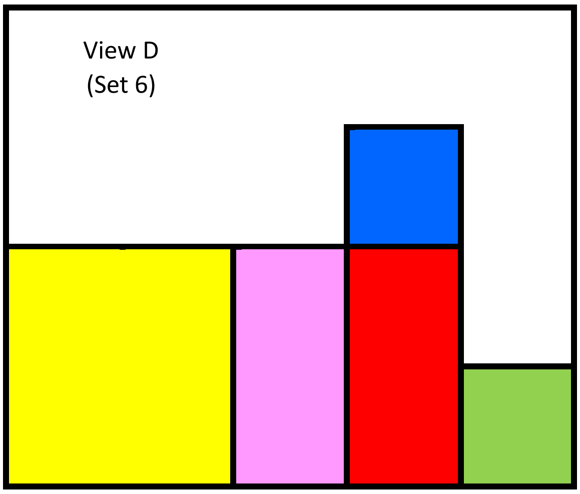


Set 6









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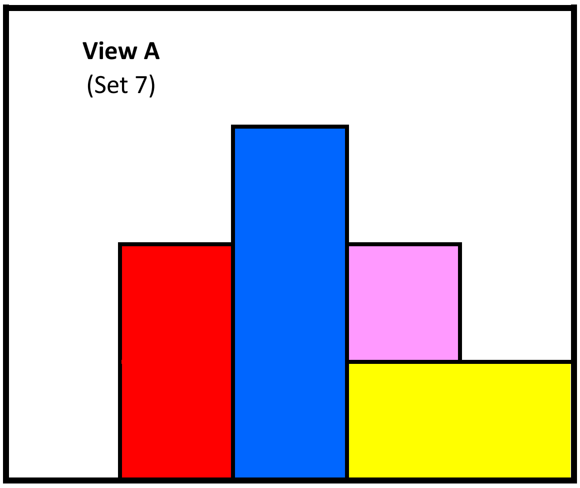
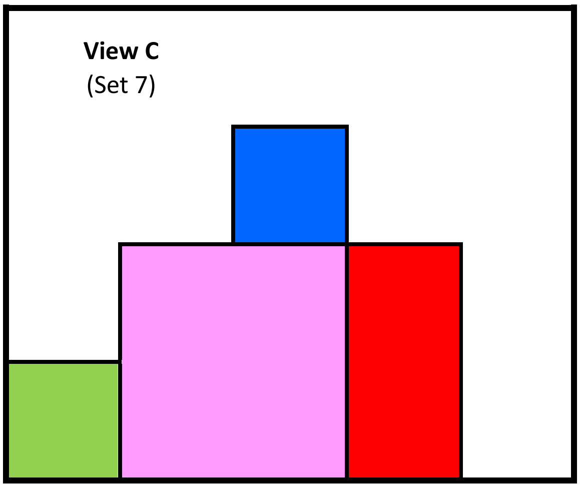
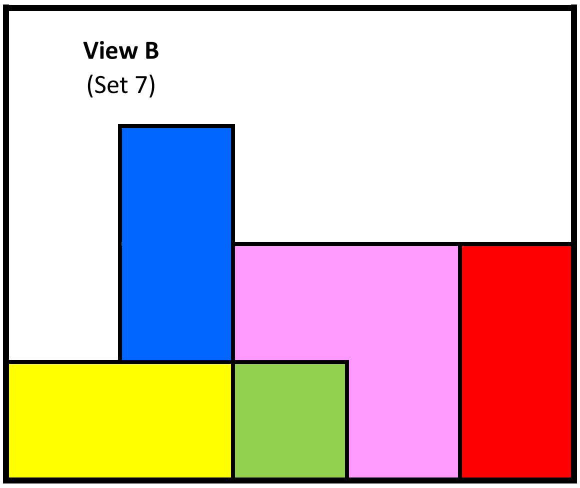
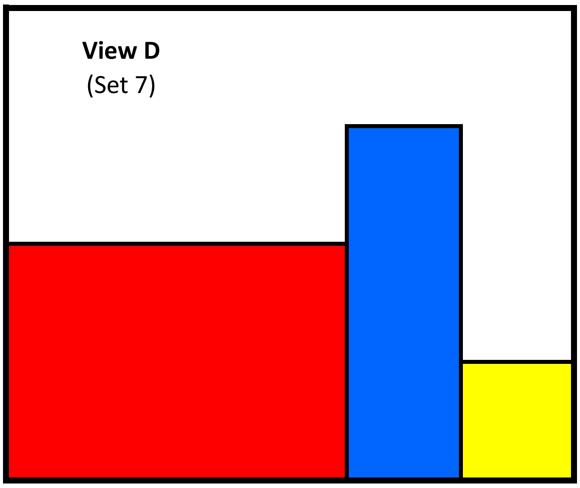








Blank View Cards

View A			

View C			

View B			

View D			

3-Dimensional constructions and properties

A GeoGenius kit consists of sturdy coloured shapes which are attached together using elastic bands. The kit includes a booklet with activity ideas. Posters are also available with 2-D pictures of objects that can be made.

Free Play Activity

During free play, learners can create any imaginative shapes.

Focused Play Activity

During focused play, teachers can ask children to build specific objects. See the Activity sheet on the following page.

Activity: Constructing 3-D objects

Use GeoGenius shapes to manufacture the beautiful objects below.

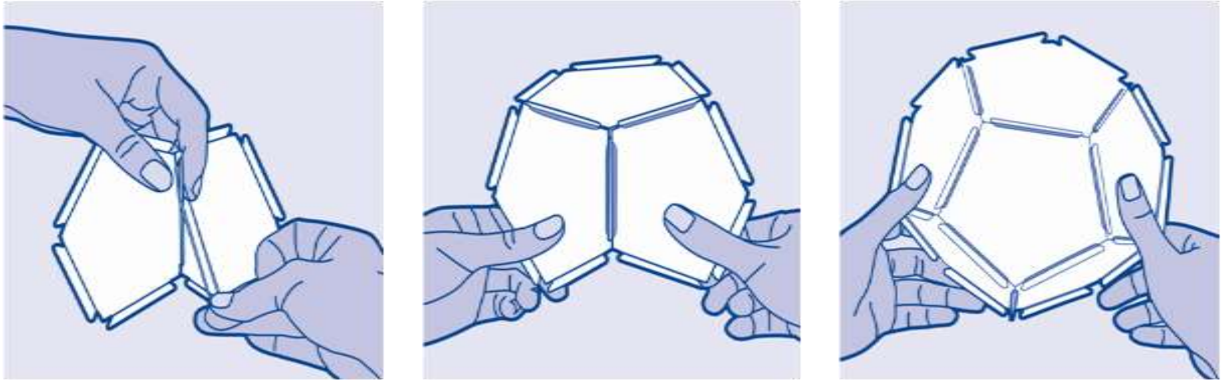
INSTRUCTIONS

STEP 1 – Fold the tabs

- Fold each of the tabs on the side of the shape along the crease by gently pushing the piece onto a solid surface such as a table.

STEP 2 – Join the pieces

- Hold the tabs of two pieces together and place an elastic band around the two tabs to join the pieces
- Repeat as required to complete the shape that you are making



Icosidodecahedron

Rhombicosidodecahedron

Rhombicuboctahedron

Truncated Icosahedron

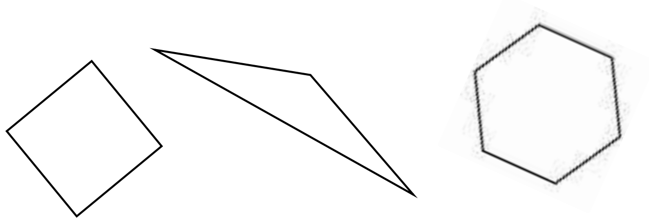
Truncated Cuboctahedron

Truncated Icosidodecahedron

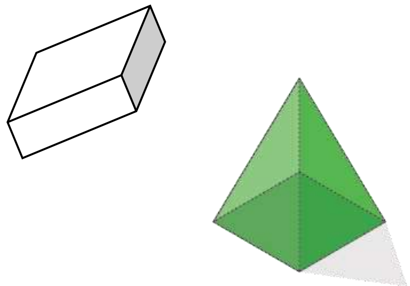
Explicitation Activity

During explication, children are introduced to some of the following terminology:

- Polygon
  - 2-D Shape
    - Triangle
    - Square
    - Hexagon
    - Octagon
    - Decagon



- Polyhedron
  - 3-D Object
    - Pyramid
    - Prism
    - Anti-prism
    - Platonic Solids
    - Archimedean solids

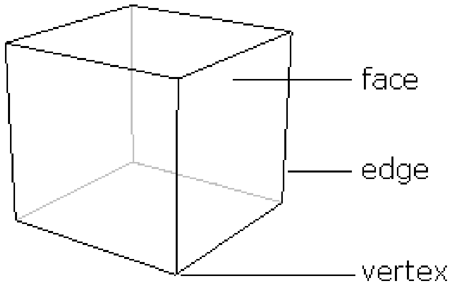


- Prisms – An object with identical parallel polygon bases and other faces are parallelograms.

- Pyramid – An object with a polygonal base whose other faces are all triangles which meet at a point called the apex.

- Anti-prisms – An object with identical parallel polygonal bases joined by an alternating band of triangles.

- Faces, edges and vertices



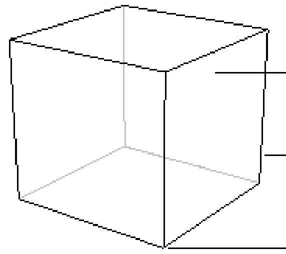
Investigation 1 and 2 can be done as Explicitation Activities

Should you wish to purchase a GeoGenius Construction kit, please refer to the Resource list.

Investigation 1: Faces, edges and vertices

Use the GeoGenius kit pieces provided to make each of the following shapes:

- Triangular based: prism, pyramid and antiprism
- Square based: prism, pyramid and antiprism
- Pentagonal based: prism, pyramid and antiprism
- Hexagonal based: prism, pyramid and antiprism



1. Refer to the objects that you have made to complete as much of each of the tables below.
2. By looking for patterns complete as much of the remainder of each table as you can.

Number of edges in the base

Prisms	3	4	5	6	7	8	9	10	*n*
Faces									
Edges									
Vertices									

Pyramids	3	4	5	6	7	8	9	10	*n*
Faces									
Edges									
Vertices									

Antiprisms	3	4	5	6	7	8	9	10	*n*
Faces									
Edges									
Vertices									

Investigation 2: Platonic solids

- A regular polygon is a shape with all sides equal in length and all angles equal in size.
- A Platonic solid is the name given to an object that has identical regular shapes (polygons) for all of its faces and there are the same number of faces around each vertex.

Build as many Platonic solids as possible.

Start with equilateral triangles.

- Can you build a polyhedron with 2 triangles around each vertex?
- Can you build a polyhedron with 3 triangles around each vertex?
- Can you build a polyhedron with 4 triangles around each vertex?
- Can you build a polyhedron with 5 triangles around each vertex?
- Can you build a polyhedron with 6 triangles around each vertex?

Continue by investigating squares, pentagons, hexagons, etc.

How many Platonic solids can you build?

GeoGenius CONSTRUCTION KIT

Making nets of 3-D objects using GeoGenius kits

The *GeoGenius construction kit* is a powerful tool that can be used to design the nets of objects.

Instructions:

- Step 1: Make the object that you want to design a net for using the pieces of the *GeoGenius construction kit*
- Step 2: Carefully remove elastic bands from the object one by one until you can completely flatten the entire surface of the object – make sure not to remove so many elastics that the pieces come lose.
- Step 3: Make a rough sketch of the net of the object
- Step 4: Make an accurate drawing of the net on a piece of light cardboard – you might want to use the pieces from the *GeoGenius construction kit* to trace around but leave off the tabs as you do so
- Step 5: Add glue tabs to the net that you have created
- Step 6: Cut out the net and crease (score) the fold lines of the net by drawing along them with a ball-point pen
- Step 7: Assemble and glue the object that you have made.

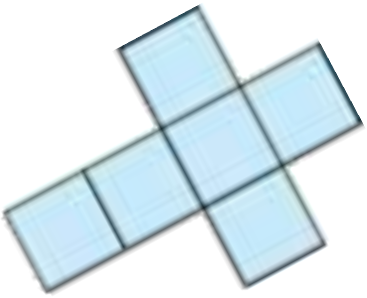
Focused Play

Children can be asked to draw nets of 3-D objects as described above.

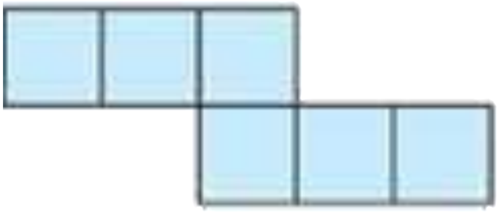
Children can also investigate different types of nets. See the investigation on the following page.

Investigation 3: Nets of cubes

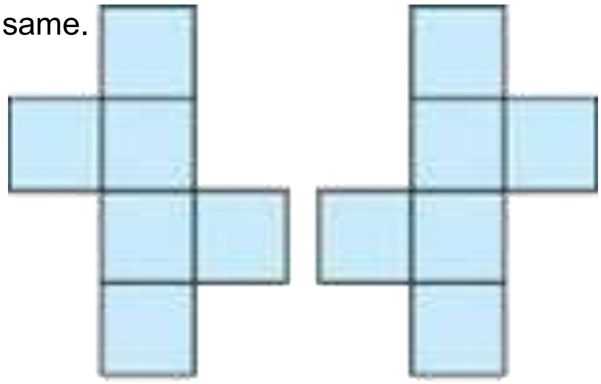
- This is a net of a cube:



- Is this a net of a cube?



- These nets are the same.



- Use the GeoGenius pieces to investigate how many <u>different</u> nets of cubes there are. Make a sketch of each unique net that you find.

Perimeter, area, volume and π

<u>Definitions</u>

Perimeter: Perimeter refers to the total distance around the outside of a 2-D figure.

Area: Area refers to the number of square units of a certain size needed to cover the surface of a figure.

<u>Activity</u>: *Refer to the following four investigations and task*. Investigations can be done in groups.

<u>Resources</u>: These investigations and the task should be copied for each learner. Learners will also need:

- Plain (unlined) paper, about three pieces.
- Two pieces of A4 coloured cardboard.

Investigation 1

In this investigation we will use construction and measuring instruments to estimate the value of π. Pi is an irrational number that relates the circumference of a circle to its diameter.

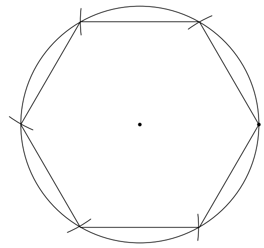
- You should work in groups of three to four people with each person choosing a different value for the diameter of the circle that you are going to work with. Choose from:
  - 8 cm
  - 10 cm
  - 12 cm
  - 14 cm

- Follow the instructions that follow and use the values that you determine to complete the table below for your own values:

Diameter of the circle being used by the investigator: __________cm				
Interior polygon			Exterior polygon	
perimeter (cm)	ratio perimeter : diameter		ratio perimeter : diameter	perimeter (cm)
		stage 1		
		stage 2		
		stage 3		

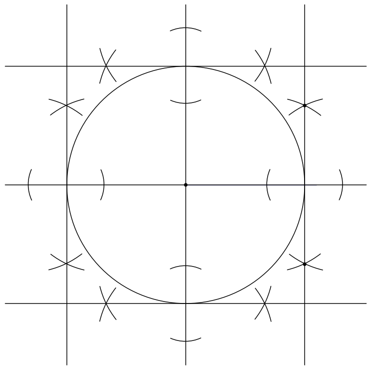
Stage 1

On a plain piece of paper construct a circle with the diameter that you have been allocated and then:



- Use a pair of compasses to create a regular hexagon inside the circle as follows:
  - Place the point of the pair of compasses anywhere on the circumference of the circle and mark the length of the radius of the circle on the circle
  - Place the point on the pair of compasses on the mark created above and again mark the length of the radius of the circle on the circle.
  - Repeat until you have six equally spaced points on the circumference of the circle.

- Use a pair of compasses and a ruler to create a square outside the circle as follows:
    - o Mark a point on the edge of the circle and draw a diameter through this point extending it beyond the circumference of the circle.
    - o Construct perpendicular lines to the diameter through the end points of the diameter and the centre of the circle using your pair of compasses and a ruler.
    - o Construct perpendiculars through the end points of the perpendicular diameter created above and complete the square

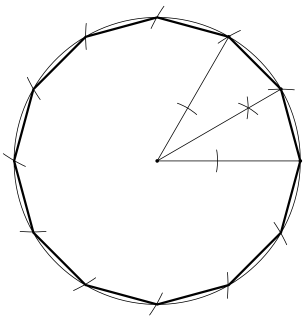


- Use your ruler to measure the perimeter of both the interior hexagon and the exterior square. Enter these values into the table on page 1.

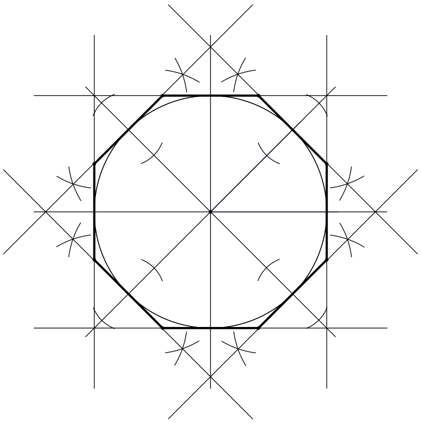
Stage 2

Next construct a circle with the diameter that you have been allocated and then:

- Use a pair of compasses to create a regular 12 sided polygon inside the circle as follows:
    - o Place the point of the pair of compasses anywhere on the circle and mark the length of the radius of the circle on the circle.
    - o Construct the angle through the two points and the centre of the circle and bisect this angle extending the angle bisector to cut the circle.
    - o Using your pair of compasses mark the length from the point on the circumference to the point where the angle bisector meets the circumference off on the circle twelve times and construct the regular 12 sided polygon



- Use a pair of compasses and a ruler to create a regular octagon outside the circle as follows:
    - o Make a square outside the circle as before.
    - o Join the centre of the circle to the vertices of the square and where the diagonals of the square intersect the circle make lines that are perpendicular to the diagonals and create another square.
    - o Mark the intersections of the two squares and complete the regular octagon



- Use your ruler to measure the perimeter of both the interior hexagon and the exterior square. Enter these values into the table on page 1.

Stage 3

Finally construct a circle with the diameter that you have been allocated and then based on your experience of stages 1 and 2:

- Construct a regular 24 sided polygon on the inside of the circle.
- Construct a regular 16 sided polygon on the inside of the circle.
- Use your ruler to measure the perimeter of both the interior hexagon and the exterior square. Enter these values into the table on page 1.

What can you say about the circumference of the circle in terms of the diameter of the circle?

Task 1

Had your constructions been precise you would have established the following ratios:

Interior polygon			Exterior polygon	
polygon	ratio perimeter : diameter		ratio perimeter : diameter	polygon
regular hexagon	3,00	stage 1	4,00	square
regular 12-sided	3,11	stage 2	3,31	regular octagon
regular 24-sided	3,13	stage 3	3,18	regular 16-sided

Consider a circle with a radius of 1 m. Now calculate the perimeter of the circle using the ratios determined at each of the stage above and reflect on how precise/imprecise these estimates for pi are.

Investigation 2

Use the grid paper below to draw at least four different rectangles with an area of 36 square units. Then complete the table below:

Reflection

What relationship do you notice between the length, breadth and area of the rectangles? Discuss how this relationship can be used to determine the area of any rectangle.

Rectangle	Length	Breadth	Area
1			
2			
3			
4			

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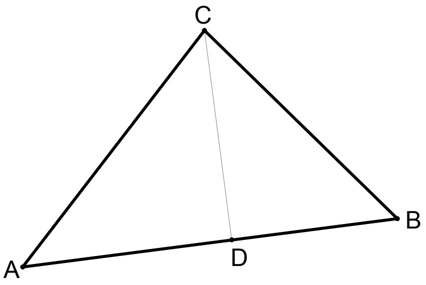
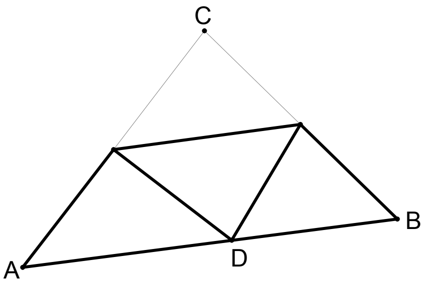
Investigation 3

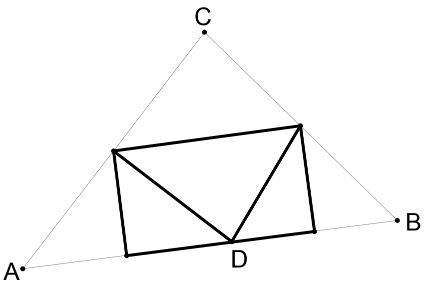
Use coloured cardboard and the instructions below to determine a formula that could be used to determine the area of a triangle:



Instructions:

- Cut out a triangle on coloured cardboard. Trace around the triangle in the space provided below.
- Fold the triangle through the point C to the point D so that the edge BD lies on top of AD.
- Fold the point C down to the point D
- Fold the points B to D and A to D.





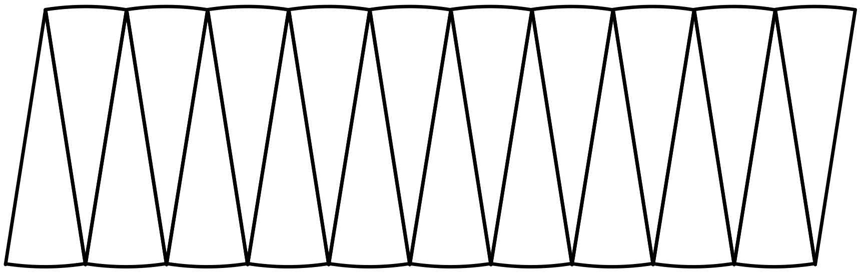
Stick the folded triangle on top of the traced triangle and use your knowledge of the area of a rectangle to create a formula that can be used to determine the area of a triangle

Investigation 4

Use the coloured cardboard provided and the instructions below to determine a formula that could be used to determine the area of a circle:

Instructions:

- Cut out the circle that you have been given.
- Cut the circle in half along one of the lines and swop with a neighbour so that you have a half circle of each colour.
- Cut out the segments of each of your half circles.
- Glue the segments down as illustrated alongside.



Use your knowledge of the area of a rectangle to create a formula that can be used to determine the area of a triangle.

Print templates for investigations 3 and 4