Manipulating Numbers

An extract from the NumberSense Mathematics Programme ***Counting, Manipulating Numbers and Problem Solving*** Booklet, which can be downloaded from **www.NumberSense.co.za**

A_Unit E23, Prime Park, Mocke Road, Diep River | T_ (021) 706 3777 | E_info@NumberSense.co.za | **www.NumberSense.co.za**

Developing a Sense of Number in the Early Grades

Contents

Developing a Sense of Number

Background

As children develop their sense of number there are clearly identifiable stages/milestones: *counting all*, *counting on*, and *breaking down and building up numbers* (*decomposing, rearranging, and recomposing*).

When we observe children at work with numbers – in particular: solving problems with numbers – we can tell what stage of number development they are at.

- If we ask a child to calculate 3 + 5 and we observe that she "makes the 3" and "makes the 5" (using fingers or objects) before she combines the objects and counts all of them to determine that 3 + 5 = 8, then we say that this child is at the *counting all* stage.
- If we observe the child becoming more efficient by "making" only one of the numbers (using fingers or objects) and then counting these objects on from the other number "5: 6, 7, 8" to conclude that: 5 + 3 = 8, then we say that this child is at the *counting on* stage.
- If we observe a child, manipulating numbers to make the calculation easier, for example by saying that 8 + 7 = 8 + 2 + 5 = 10 + 5 = 15, we say that she has reached the *breaking down and building up* stage. What the child has done is to break up one of the numbers: 7 into 2 and 5 which allows her to "complete the 10" by adding the 2 to the 8 and then adding the remaining 5 to the 10 to get 15. We refer to this stage (more formally) as the *decomposing, rearranging and recomposing* stage.

It is expected that all children should reach the *breaking down and building up* stage within age appropriate number ranges.

In the early grades, we support children's development of the number concept through three distinct but interrelated activities:

- Counting,
- **Manipulating numbers**, and
- Solving problems

Counting (in full version)

Manipulating numbers

Calculating fluently and efficiently

Numbers and calculations with numbers are at the heart of mathematics. Children need to develop a range of calculating strategies that enable them to calculate flexibly and fluently. Furthermore, it is important that they can perform a wide range of calculations mentally. Mental calculations are central to estimating.

It is unlikely that we would expect anybody to spend time calculating 24.382 × 0.248 using paper and pencil in a context where we have calculators. However, it is important that a person has a sense of what the expected answer is. In the case of 24.382 × 0.248 we expect a person to have a sense that 24.382 × 0.248 ≈ 12 × 0,5 = 6 so that when they use their calculator and get 6.046736 as the answer they are not surprised.

Calculating flexibly means using different calculation strategies for different situations.

Calculating fluently means confidently using a range of calculation strategies within a number range and for operations appropriate to a child's developmental state.

To illustrate flexibility consider the two calculations:

$$37 + 49 = \boxed{} \text{ and } 36 + 47 = \boxed{}$$

When performing the first calculation mentally it may make it easier to think of the calculation as:

$$37 + 49 = 37 + 50 - 1 = 87 - 1 = 86$$

This approach involves recognising that 49 is very close to 50. Adding 50 to 37 is easy, so what remains is to subtract the 1 that was added to 49 to create 50.

When performing the second calculation mentally it may make it easier to think of the calculation as:

$$36 + 47 = 36 + 4 + 43 = 40 + 43 = 83$$

This approach involves breaking up the 47 into 4 + 43. This is done because 4 is needed to "fill up the 36 to make 40," and, adding 40 to the remaining 43 is quite easy.

The illustration makes the point that the calculation strategy used to perform a calculation is chosen in terms of the numbers being calculated with. Flexibility in calculating refers to an individual's ability to *choose* an effective strategy for the calculation being performed. It goes without saying, that the calculation strategy used by a child is determined by the calculation being performed, as well as the child's developmental state, the child's confidence and their "sense of number".

Almost all calculation strategies involve breaking down, rearranging and building up numbers. We break down one or more of the numbers in a way that will make the calculation easier. We rearrange the numbers that we now have and then we build up the result. In the case of 36 + 47 = $\square$:

- We broke up 47 into 43 + 4 because the 4 would help us to make 36 into 40,
- Rearranged the numbers: 36 + 4 + 43, and
- Built up the resulting 83 by first adding 36 + 4 and then adding the remaining 43.

In order for children to be able to calculate flexibly and fluently they need to develop a wide range of different number manipulation and calculating strategies. At the same time they need to have a great deal of practice in using these strategies.

We help children begin to develop the different manipulation and calculation strategies through daily mental arithmetic activities. These daily mental arithmetic activities should reveal the strategies through patterns. Daily mental arithmetic activities should be part of our daily classroom routine.

NOTE: Development of the different number manipulation and calculation strategies are the consequence of deliberately designed and carefully coordinated classroom activities. We refer to these as the **number manipulation activities**.

A ***mental number line*** is at the heart of mental arithmetic and calculating flexibly and fluently. A mental number line is an image in the mind of a number line that children move up and down with confidence. At first a child's number line will include only the single digit numbers. With time, however, the number line will 'grow' and children will gain confidence in moving up and down the line focusing on different parts. They will be able to 'zoom out' and see the number line stretching from 0 to 100 and they will be able to 'zoom in' to see all the fractions between 5 and 6. As children gain confidence in moving around the number line they will begin to notice that:

- In the same way that 6 is one more than 5:
    - 26 is one more than 25, and
    - 146 is one more than 145.
- In the same way that 10 is 3 more than 7:
    - 30 is 3 more than 27, and
    - 270 is 3 more than 267.
- In the same way that 4 + 5 = 9:
    - 40 + 50 = 90, and
    - 400 + 500 = 900.
- In the same way that 8 + 7 is the same as 8 + 2 + 5 = 10 + 5 = 15:
    - 68 + 7 is the same as 68 + 2 + 5 = 70 + 5 = 75, and
    - 285 + 79 is the same as 285 + 15 + 64 = 300 + 64 = 364.

We nurture the development of a child's mental number line as well as their ability to calculate fluently and flexibly through the careful use of deliberately structured activities that develop the child's confidence with:

- Single digit arithmetic,
- Arithmetic with multiples of 10, 100 and 100,
- Completing 10s, 100s and 1 000s,
- Bridging 10s, 100s and 1 000s,
- Doubling and halving, and
- A wide range of multiplication facts.

To many people, these are the so-called "basic number facts". The "number facts" that children should know in order "to do mathematics". It is true that children who do not have access to these number manipulation and calculation skills are unlikely to develop fluency and flexibility with calculating. However, it is not true that memorising the "basic number facts" means that children can apply them. Children need to know their "basic number facts" in an interrelated and integrated way. They need to "see" the patterns. Modern cognitive science talks about 'constellatory thinking'.

Recent literature on learning mathematics talks about the need for *Procedural Fluency* as one dimension of so-called *Mathematical Proficiency*. Procedural fluency is not to be confused with memorisation. Procedural fluency does, however, involve the automatisation of tasks. There comes a point in every child's mathematical life when they "know" that 5 + 3 = 8. They know it without first 'making' five and 'making' three and putting them together and counting eight. Children reach this state of 'automaticity' through frequent interaction with these mathematical relationships. For this reason, daily practice is important and the need for frequent structured manipulating number activities self-evident.

General description of manipulating numbers activities

The teacher typically works with one group of children at a time. The teacher arranges small groups of children (8 to at most 10) in a circle on the mat. The teacher should be a part of the circle, so that he/she can make eye contact with each child. As children become more confident and familiar with the activity the teacher can increase the size of the group and eventually conduct the activity with the whole class.

<u>Activity</u>

- The teacher working with the group of children:
  - Tells them to put down their pencils and to sit quietly.
  - Reminds them that they should try not to use their fingers when working out the answers to the questions,
  - Asks them not to shout out the answers to her questions and to wait to be asked to give their answer before doing so, and

    - To calculate the answers to all questions even if the question is not directed at them. They may be asked for their answer if the child that the teacher asked struggled with the question or gets the answer wrong.
- The teacher then goes around the group randomly selecting children to answer her questions.
    - There are typically two ways of asking a question:
        - As a direct calculation:
            - "What is 5 plus 4?"
            - "What is half of 36?"
            - "What is double 14?"
            - "What is 167 minus 50?"
        - As an equation to be solved:
            - "What must be added to 7 to get 10?"
            - "What must taken away from 45 to be left with 38?"
            - "What must be doubled to give 72?"
- The teacher structures the questions in sets to reveal the patterns that she wants children to observe. For example:
    - When *adding and subtracting with multiples of 10, 100 and 1000*, the teacher may ask:
        - "What is 5 + 2?"
        - "And what is 500 + 200?"
        - "And what is 5000 + 2000?"
        - "And 50 + 20?"
        - "What did you notice?"
        - "What do you think 400 + 300 will be? Why do you say that?"
    - When *completing 10s and 100s*, the teacher may ask:
        - "What is 7 + 3?"
        - "And what is 27 + 3?"**
        - "And what is 67 + 3?"
        - "What must be added to 87 to get 90"
        - "And what must be added to 147 to get 150?"
        - "What did you notice?"
        - "What do you think must be added to 137 to get 200? Why do you say that?"

        ** Note how 17 + 3 is initially left out. We will come back to this when children are more confident. The reason is that "twenty-seven plus three", sounds a lot more like "seven plus three" than "seventeen plus three" does. We want children to 'see' the pattern.
- The teacher adapts the number range of the questions asked to the developmental state of the children she is busy working with.

<u>A few comments about the activity</u>:

- About the difficulty level of the questions being asked by the teacher.
  - When managing the manipulating number activities teachers should bear in mind that:
    - We want children to respond quickly and confidently to the questions posed. If a child struggles then the question being asked is too difficult and the teacher needs to first try an easier version of the question.
    - Children should be discouraged from using their fingers. If the teacher notices children using their fingers then she knows that the question being asked is too difficult. If this is the case the teacher needs to try an easier version of the question.
- Throughout the activity teachers should ask children to explain how they performed a calculation; especially if a particular child answered a question with confidence or very quickly.
    - Often the teacher will notice that if a child has 'seen the pattern' and is asked to explain how they did the calculation so confidently, they will be encouraged to reflect on their thinking and to articulate it. Reflecting on and describing their thinking helps children to learn.
    - By asking a child to articulate what they did when they answered a question will not only help their thinking, but also help the other children in the group to develop their understanding. To support the other children the teacher can ask one of the children who listened to the explanation:
      - "Do you understand what your friend just said?"
        - If yes, "Can you illustrate what he did by solving this problem?" and then asking the child a question with a similar structure.
        - If no, "Shall we ask him to explain it again?"
      - "Can you use what your friend did to solve the following problem?" and then asking the child a question with a similar structure.
- Teachers often make the mistake of thinking that if they "completed 10s" today and some children saw the pattern then they know how to complete tens and don't need to revisit/practice it again. This is not the case. Children need sustained practice and therefore need to be asked the same questions again and again on a regular basis. Number manipulation needs to be a daily classroom activity and needs to last for at least 5 to 10 minutes per day.
- It is not enough that children are able to do 'single digit arithmetic', 'complete' and 'bridge 10s, 100s and 100s', etc. during the daily number manipulation slot. They also need to use it in the calculations they do and problems they solve in the rest of the mathematics lesson. The teacher has an important role to play in helping children make these links. She does so by referring to these skills whenever appropriate and by example (using these skills herself).

Specific description of the different manipulating numbers activities

Manipulating numbers activity 1: Single digit arithmetic

This activity involves the addition (and subtraction) of single digits to (and from) numbers of varying sizes. These calculations do not involve the bridging of a ten (decade).

The bridging of decades is introduced in activity 3.

There are 20 fundamental addition (and subtraction) facts that all children need to know to the point of automaticity. They are:

1 + 1 = 2			
2 + 1 = 3			
3 + 1 = 4	2 + 2 = 4		
4 + 1 = 5	3 + 2 = 5		
5 + 1 = 6	4 + 2 = 6	3 + 3 = 6	
6 + 1 = 7	5 + 2 = 7	4 + 3 = 7	
7 + 1 = 8	6 + 2 = 8	5 + 3 = 8	4 + 4 = 8
8 + 1 = 9	7 + 2 = 9	6 + 3 = 9	5 + 4 = 9

The assumptions in listing these 20 facts are that if a person knows that:

- 4 + 3 = 7 then they also know that 3 + 4 = 7. This is also known as the commutative property of addition.
- 4 + 3 = 7 then they also know that 7 – 4 = 3 and 7 – 3 = 4

The exciting thing about these facts is that they are not limited to single digit addition and subtraction with totals less than or equal to 9. As we develop the mental number lines (mentioned earlier), children develop the awareness that since 4 + 3 = 7:

- 24 + 3 and 134 + 3, and
- 40 + 30; 400 + 300 and 4 000 + 3 000

All rely on the relationship: 4 + 3 = 7.

The same can be said for subtraction since 6 + 2 = 8, it follows that:

- 8 – 6 = 2; 38 – 6 = 32 and 158 – 6 = 152, and
- 80 – 60 = 20; 800 – 600 = 200 and 8 000 – 6 000 = 2 000.

<u>Questions used by teachers to develop single-digit arithmetic</u>:

1. For young children with a primitive sense of number (late Grade R and early Grade 1):

   Start with the following questions:

   - I want you to imagine the number line. Can you see it? Now look for the number 5. Can you see it? What number comes after 5?

- I want you to imagine the number line. Can you see it? Now look for the number 4. Can you see it? What number comes before 4?

Repeat these questions with different initial numbers but always asking about the number before and after that number. Be careful only to ask questions that do not result in or bridge the 10 – that will come later.

As the children's confidence increases (after some days/weeks) change the questions to:

- I want you to imagine the number line. Can you see it? Now look for the number 7. Can you see it?
    - What number comes after 7?
    - What number is two numbers after 7?
- I want you to imagine the number line. Can you see it? Now look for the number 6. Can you see it?
    - What number comes before 6?
    - What number is two numbers before 6?

Repeat these questions with different initial numbers but always asking about the numbers one, two, three etc. before and after that number. Only increase from one before and after to two before and after etc. once children no longer need to use their fingers to 'work out' the answer. Again, be careful to only ask questions that don't result in or bridge the 10 – that will come later.

2. For young children who have gained confidence with the questions in 1 (late Grade R and early Grade 1):

    Start with the following questions:

    - I want you to imagine the number line. Can you see it? Now look for the number 7. Can you see it?
        - What number do you reach when you add one to 7?
        - What must we add to 7 to reach 8?
    - I want you to imagine the number line. Can you see it? Now look for the number 3. Can you see it?
        - What number do you reach when you subtract (take away) one from 3?
        - What must we subtract from 3 to reach 2?

    Repeat these questions with different initial numbers but always asking about the number reached by adding or subtracting one from the number. Once again, be careful to only ask questions that don't result in or bridge the 10 – that will come later.

    As the children's confidence increases (after some days/weeks) change the questions by gradually increasing, the amounts added and subtracted (from one to two, and to three and so on). Be careful to only ask questions that don't result in or bridge the 10 – that will come later.

3. For young children who have gained confidence with the questions in 1 and 2 (mid Grade 1 through Grade 3 and above):

   Ask questions in sets that reveal patterns/relationships, using the 20 number facts listed at the start of this activity. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. That means, for the sets below initially the teacher will ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is 7 + 2? And what is:
    - 27 + 2
    - 37 + 2?
    - 47 + 2?
    - 87 + 2?
    - 157 + 2?
    - 287 + 2?
    - Etc.

- What must you add to 6 to get 8? And:
    - What must you add to 26 to get 28?
    - What must you add to 36 to get 38?
    - What must you add to 46 to get 48?
    - What must you add to 86 to get 88?
    - What must you add to 156 to get 158?
    - What must you add to 276 to get 278?
    - Etc.

- What is 8 - 3? And what is:
    - 28 - 3?
    - 38 - 3?
    - 48 - 3?
    - 88 - 3?
    - 158 - 3?
    - 288 - 3?
    - Etc.

- What must you subtract from to 5 to get 3? And:
    - What must you subtract from 25 to get 23?
    - What must you subtract from 35 to get 33?
    - What must you subtract from 45 to get 43?

- What must you subtract from 85 to get 83?
- What must you subtract from 155 to get 153?
- What must you subtract from 275 to get 273?
- Etc.

NOTES:

- The importance of the single digit arithmetic activities described above cannot be over emphasised! The activities that follow all rely on these single digit facts.
- Throughout these single digit arithmetic activities the teacher helps children to see the underlying pattern/structure by always starting with the basic fact and then asking the related questions. For example, even with a child whose confidence is increasing we still ask
“What is 8 – 5? And, what is 68 – 5? 148 – 5? And, 798 – 5?”

Manipulating numbers activity 2: Arithmetic with multiples of 10, 100 and 1000

Arithmetic with multiples of 10, 100 and 1000 is simply an application of the single digit arithmetic developed earlier. It is remarkable how many children do not see this relationship and for them adding 100 to 200 is considered difficult because it “involves large numbers”.

Ask questions in sets that reveal patterns/relationships using the 20 number facts listed at the start of the *single digit arithmetic* activity. A few examples of question sets are illustrated below.

- What is 4 + 3? And what is:
    - 400 + 300?
    - 4 000 + 3 000?
    - 40 + 30?
    - 40 000 + 30 000?
    - Etc.

Notice that the sequence of asking 400 + 300 and 4 000 + 3 000 deliberately come before 40 + 30. With our English vocabulary “four-hundred plus three-hundred” or “four-thousand plus three-thousand” are clearly related to “four plus three”. “Forty plus thirty” is not as clearly related to “four plus three”, because the number names don’t sound as similar.

- What must you add to 4 to get 6? And:
    - What must you add to 400 to get 600?
    - What must you add to 4 000 to get 6 000?
    - What must you add to 40 to get 60?
    - Etc.

- What is 7 - 2? And what is:
    - 700 – 200?
    - 7 000 – 2 000?

    - 70 – 20?
    - Etc.

- What must you subtract from to 8 to get 5? And:
    - What must you subtract from 800 to get 500?
    - What must you subtract from 8 000 to get 5 000?
    - What must you subtract from 80 to get 50?
    - Etc.

NOTE:
- As children gain confidence with this activity it should be possible to go straight to 800 + 100 etc. without first asking 8 + 1.

Manipulating numbers activity 3: Completing 10s, 100s and 1000s …

We now introduce a further 5 fundamental addition (and subtraction) facts that all children need to know to the point of automaticity. They are:
1 + 9 = 10
2 + 8 = 10
3 + 7 = 10
4 + 6 = 10
5 + 5 = 10

The assumption with these 5 facts is that if a person knows that:
- 3 + 7 = 10 then they also know that 7 + 3 = 10. This is known as the commutative property of addition.
- 3 + 7 = 10 then they also know that 10 – 7 = 3 and 10 – 3 = 7

Multiples of 10 (100 and 1 000) are very helpful interim targets in a calculation. This is because addition with multiples of 10, 100 and 1 000 is actually quite easy if it is seen as an application of single digit facts.

Remember the illustration at the start of this section: 36 + 47 = $\boxed{}$. We added 4 to 36 so that we would get to 40 because adding 40 to 43 (what is left after taking 4 from 47) is really quite easy. We refer to the process of getting to a multiple of 10 as ‘completing the 10’.

As always, we ask questions in sets that reveal patterns/relationships using the 5 new number facts listed. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. For the sets below that means that initially the teacher will just ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is 6 + 4? And what is:
    - 7 + 3?
    - 3 + 7?
    - 2 + 8?
    - Etc.

- What must you add to 6 to get 10? And:
    - What must you add to 7 to get 10?
    - What must you add to 3 to get 10?
    - What must you add to 2 to get 10?
    - Etc.

- What is 8 + 2? And what is:
    - 18 + 2?
    - 28 + 2?
    - 78 + 2?
    - 238 + 2?
    - 468 = 2?
    - Etc.

- What must you add to 3 to get 10? And:
    - What must you add to 23 to get 30?
    - What must you add to 63 to get 70?
    - What must you add to 383 to get 390?
    - What must you add to 993 to get 1 000?
    - Etc.

- With older, more confident children it should be possible to ask questions such as:
    - What must you add to 3 to get 10? And:
        - What must you add to 23 to get 30?
        - What must you add to 53 to get 70?
        - What must you add to 123 to get 200?
        - What must you add to 253 to get 1 000?
        - Etc.

      In the case of the last few questions we are expecting children to answer the questions in a few steps (with time they will not even need the steps). For example
        - What must you add to 53 to get 70?
            - I must add 7 to get to 60 and then another 10 to get to 70, the answer is 17 (10 + 7).

- What must you add to 123 to get 200?
    - I must add 7 to get to 130 and then another 70 to get to 200, the answer is 77 (70 + 7).
- What must you add to 353 to get 1 000?
    - I must add 7 to get to 360 and then another 40 to get to 400, and finally another 600 to get to 1 000, the answer is 647 (600 + 40 + 7).

NOTE:

- The last few examples illustrate the importance of being able to add numbers like 600, 40 and 7 to get 647. More generally it is just as important to be able to add 34 and 40 to get 74. This is referred to as adding to and subtracting from multiples of 10. Adding to and subtracting multiples of 10 includes examples like:
    - 10 + 7 = 17, 40 + 7 = 47 and 400 + 7 = 407
    - 17 – 10 = 7, 57 – 30 = 27 and 277 – 30 = 247
    - 34 + 40 = 40 + 34 = 74 and 50 + 36 = 86
    - 240 + 34 = 274
    - 284 – 30 = 254

    In these notes we do not make a special case for this category of questions. If children know that 254 = 200 + 50 + 4, and they are confident with arithmetic using single digits and multiples of 10, 100 and 1000 then they do not experience the questions illustrated above as special.

Manipulating numbers activity 4: Bridging 10s, 100s and 1000s

When children are confident with *single digit arithmetic*, *arithmetic with multiples of 10, 100 and 1 000*, and *completing 10s, 100s and 1 000s*, they are ready for *bridging 10s, 100s and 1 000s*.

The process of bridging 10s is best illustrated by an example. Consider the addition problem 8 + 7. Children who are confident with the skills already developed will think of this problem as follows:

- What do I need to add to 8 to make 10? … 2.
- Let me break 7 into 2 + 5.
- Then 8 + 7 is the same as 8 + 2 + 5 = 10 + 5 = 15

In time, we expect children to reach a level of automaticity where they don't have to do detailed thinking when adding/subtracting single digit numbers up to 9 + 9 = 18 and 18 – 9 = 9. However, they reach this level of automaticity only after thinking about the problem as illustrated above.

Let us illustrate this using a few more examples:

- What is 36 + 7?
    - 36 + 4 + 3 = 40 + 3 = 43
- What is 56 + 38?
    - 56 + 4 + 34 = 60 + 34 = 94

- What is 87 + 56?
    - 87 + 3 + 53 = 90 + 53 = 90 + 10 + 43 = 100 + 43 = 143
- What is 687 + 568?
    - 687 + 3 + 565 = 690 + 10 + 555 = 700 + 555 = 700 + 300 + 255 = 1 000 + 255 = 1 255
  - Or, as children gain confidence:
    - 687 + 13 + 555 = 700 + 555 = 700 + 500 + 55 = 1 200 + 55 = 1 255

The same process applies to subtraction. This is also best illustrated using a few examples:

- What is 35 – 8?
    - 35 – 5 – 3 = 30 – 3 = 27
- What is 56 – 38?
    - 56 – 6 – 32 = 50 – 32 = 50 – 30 – 2 = 20 – 2 = 18
- What is 564 – 387?
    - 564 – 4 – 383 = 560 – 60 – 323 = 500 – 300 – 23 = 200 – 20 – 3 = 180 – 3 = 177
  - Or, as children gain confidence:
    - 564 – 64 – 323 = 500 – 320 – 3 = 180 – 3 = 177

The most important observation to be made in each of illustrations above is that the thought process involves the breaking up of numbers in order to complete 10s. Breaking up of numbers follows from single digit arithmetic. This is why the activities for single digit arithmetic and completing 10s are important. They are the foundational skills of addition and subtraction involving the bridging of 10s (decades).

Ask questions in sets that reveal patterns/relationships. It is also important to ask children to explain their thinking after they respond to your questions. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. For the sets below that means that initially the teacher will just ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is 6 + 4? And what is:
    - 6 + 7? Explain how you worked that out.
    - 26 + 7? Explain how you worked that out.
    - 126 + 7? Explain how you worked that out.
    - Etc.

- What must I add to 7 to get 10? And what is:
    - 7 + 5? Explain how you worked that out.
    - 37 + 5? Explain how you worked that out.
    - 47 + 15? Explain how you worked that out.
    - 267 + 5? Explain how you worked that out.

  - 267 + 25? Explain how you worked that out.
  - Etc.

- What is 10 – 2? And:
  - 15 – 7? Explain how you worked that out.
  - 35 – 7? Explain how you worked that out.
  - 35 – 17? Explain how you worked that out.
  - Etc.

NOTES:

- Children who are able to calculate in the way illustrated in this activity have reached a level of fluency and flexibility with number and operations with numbers. This makes them efficient in calculating with numbers and frees them to think about the mathematics they are doing without having to spend their energy on the manipulation of numbers.
- The level of confidence illustrated in the examples above is reached only after a long time and after deliberate and frequent practice with the activities listed above.

Manipulating numbers activity 5: Doubling and halving

Doubling and halving is at the heart of efficient multiplication and division strategies. Children who can double and halve with confidence are able to multiply and divide with understanding and efficiency.

Doubling and halving literally refers to the doubling of numbers: double 15 is 30, and halving of numbers: half of 64 is 32.

If we ask a young pre-school child to share a pile of 18 counters, equally among three dolls they will usually do something like this:

- First they will give each doll 3 counters.
- After looking at the remaining pile of counters they may realise that they do not have enough counters to give each doll another three counters and so they will give each doll two counters.
  - It is as if they have "halved" the number they gave each doll the first time around. "Halving" in this case meaning a smaller pile.
- Again, they will look at the remaining pile of counters and notice that this time they do not have enough counters to give each doll another two counters. They will then give them one each (another "halving") and there will be no counters left.

In this illustration we notice that the child gives a large amount to each doll and then a smaller amount and another smaller amount until there is nothing left over. The child is using a process of repeated "halving".

Consider another example: 7 845 ÷ 23 (7845 marbles shared by 23 people). One way of doing this calculation might involve the following thought process:

- Let's start by giving each of the 23 people 100 marbles each. That is 2 300 marbles.
- We are left with 7 845 – 2 300 = 5 545 marbles to share out.
- Double 2 300 is 4 600. 5 545 is more than 4 600, so let us give each of the 23 people another 200 marbles.
- We are left with 5 545 – 4 600 = 945 marbles.
- Double 46 is 92, and double 460 is 920. If we give everybody another 40 marbles, we would use 920 marbles (which is less than 945).
- We are left with 945 – 920 = 25 marbles enough to give everybody another one. And, there will be two marbles left over.
- It follows that 7 845 ÷ 23 = 100 + 200 + 40 + 1 → 341 remainder 2.

When the solution is written like this it appears very clumsy, however when it is written on paper as follows it makes a lot more sense:

$$
\begin{array}{rr}
7\,845 & \\
-\ \underline{2\,300} & 100 \\
5\,545 & \\
-\ \underline{4\,600} & 200 \\
945 & \\
-\ \underline{920} & 40 \\
25 & \\
\underline{23} & +\ \underline{\ \ 1} \\
 & \\
2 & 341
\end{array}
$$

7 845 ÷ 23 = 341 remainder 2

The point of these illustrations is to make the case for the value of doubling and halving in solving problem situations involving division and multiplication.

Ask questions in sets that reveal patterns/relationships. It is also important to ask children to explain their thinking. A few examples of question sets are illustrated below. The only difference between the questions asked to children at different stages of development and confidence is in the size of the numbers in the questions. For the sets below that means that initially the teacher will just ask the first few questions in each set and with time she will ask all of the questions in the set.

- What is double 4? And what is:
    - Double 30?
    - Double 34? Explain how you worked that out.
    - Etc.

- What is double 7? And what is:
  - Double 60?
  - Double 67? Explain how you worked that out.
  - Etc.
- What is half of 8? And what is:
  - Half of 20?
  - Half of 28? Explain how you worked that out.
  - Etc.
- What is half of 60? And what is:
  - Half of 10?
  - Half of 70? Explain how you worked that out.
  - Etc.

Manipulating numbers activity 6: Multiplication facts

Multiplication tables are another example of so-called “basic facts” that children must know. As with the single digit addition facts discussed earlier it is indeed important that children reach a level of automaticity with respect to multiplication. However, in the same way that the case was made earlier, it is much more important to think about how children know their multiplication facts and reach levels of automaticity than it is for them to memorise these facts.

Let us say for now that we want children to know their tables up to 12 x 12; this is (as the discussion will show) quite artificial, but let us work with this for now. If children have to remember/recite/memorise their multiplication tables then there are 144 facts to be remembered.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

If we now agree that the one-times table is trivial then there remain 131 ‘facts’.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

If we then know the commutative law of multiplication, namely that 3 x 5 = 5 x 3, then we reduce the number of facts to be remembered to 66 (already an improvement on 144):

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

a × b = b × a

Now we are ready to start “our tables”. We should start with the 10 x table. Why? Because it is so easy! Furthermore, when you see the pattern you are no longer limited to 1 x 10 to 12 x 10, in fact 43 x 10 and 284 x 10 are completely accessible to a Grade 3 child (even many Grade 2 children) who ‘see the pattern’:

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

a × b = b × a

The beauty about the 10× table is that if you know the 10× table and you know how to halve then the 5× table follows. 7 × 5 is half of 7 × 10 = 70 ➝ 35. Again, we are no longer limited to 1 × 5 to 12 × 5 because 43 × 5 and 284 × 5 are completely accessible to a Grade 3 or 4 child who 'sees the pattern':

- 43 × 5 = half of 43 × 10 = half of 430 ➝ which is half of 400 (200) and half of 30 (15) = 215.
- 284 × 5 = half of 284 × 10 = half of 2840 ➝ which is half of 2800 (1400) and half of 40 (20) = 1420.

And now it gets more exciting, because if you know your 10× table and the 5× table follows through halving then you also get the 15× table. For example:

- 43 × 15 = 430 (43 × 10) + 215 (43 × 5 = half 43 × 10 - 430) = 645.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	a × b = b × a						72	81	90	99	108
10	10							80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Next, children should know their 2× table – which is really no more than doubling. In other words doubling is the important skill (just as halving was for the 10 and 5 times table). If you can double you can multiply by 2 and by 4 (doubling again) and by 8 (doubling again). Once more if a child understands multiplying by 2 (4 and 8) as doubling they are no longer limited to 1 × 2 to 12 × 8. For example:

- 24 × 8 ➝ 24 doubled ➝ 48 doubled ➝ 96 doubled again ➝ 192.

Now notice how we are down to 21 facts still to be remembered:

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	a × b = b × a						72	81	90	99	108
10	10							80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Next children should 'learn':

- The 9× table: multiplying by 9 is like multiplying by 10 and taking one away
- The 11× table: multiplying by 11 is like multiplying by 10 and adding one on.

Notice we are now down to 10 multiplication facts still to be remembered:

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

a × b = b × a

If children then learn the 3× table, the 6× table follows through doubling and they have mastered the 144 facts we started out with and more.

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	72	81	90	99	108
10	10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

a × b = b × a

Solving Problems (in full version)