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Developing Geometric Thinking through Activities That Begin with Play

Pierre M. van Hiele

For children, geometry begins with play. Rich and stimulating instruction in geometry can be provided through playful activities with mosaics, such as pattern blocks or design tiles, with puzzles like tangrams, or with the special seven-piece mosaic shown in **figure 1.** Teachers might ask, How can children use mosaics, and what geometry do they learn? Before addressing these questions and exploring the potential of the mosaic puzzle for teaching geometry, I note some misconceptions in the teaching of mathematics and present some of my ideas about levels of thinking in geometry.

Pierre M. van Hiele, a lifelong resident of the Netherlands, is a former Montessori teacher and the author of a curriculum series that features a rich array of geometry activities. He is also world renowned for his work on levels of thinking as they relate to geometry and discusses them in this article.

During a 1987 visit to Brooklyn College, van Hiele was introduced to this mosaic puzzle. Since then, he has been fascinated with this puzzle and with the many ways that it can be used to teach geometry. Although he has lectured on activities involving the mosaic puzzle, this is the only article of which he is aware that discusses ways of using it to teach geometry concepts.

This is the first article by van Hiele to be published in a journal of the National Council of Teachers of Mathematics. For this reason, the Editorial Panel of Teaching Children Mathematics *is especially grateful to have his article appear in the journal's 1999 Focus Issue on "Geometry and Geometric Thinking." The Panel also wishes to acknowledge the work of our colleague, David Fuys, who helped van Hiele prepare the final draft of his manuscript.—Charles Geer, for the Editorial Panel.*

Misunderstandings in Teaching Mathematics

The teaching of school mathematics—geometry and arithmetic—has been a source of many misunderstandings. Secondary school geometry was for a long time based on the formal axiomatic geometry that Euclid created more than 2000 years ago. His logical construction of geometry with its axioms, definitions, theorems, and proofs was—for its time—an admirable scientific achievement. School geometry that is presented in a similar axiomatic fashion assumes that students think on a formal deductive level. However, that is usually not the case, and they lack prerequisite understandings about geometry. This lack creates a gap between their level of thinking and that required for the geometry that they are expected to learn.

A similar misunderstanding is seen in the teaching of arithmetic in elementary school. As had been done by Euclid in geometry, mathematicians developed axiomatic constructions for arithmetic, which subsequently affected the arithmetic taught in schools. In the 1950s, Piaget and I took a stand against this misunderstanding. However, it did not help, for just then, set theory was established as the foundation for number, and school arithmetic based on sets was implemented worldwide in what was commonly called the "new math." For several years, this misconception dominated school mathematics, and the end came only after negative results were reported. Piaget's point of view, which I support affectionately, was that "giving no education is better than giving it at the wrong time." We must provide teaching that is appropriate to the level of children's thinking.

Levels of Geometric Thinking

At what level should teaching begin? The answer, of course, depends on the students' level of thinking. I begin to explain what I mean by levels of thinking by sharing a conversation that two of my daughters, eight and nine years old at the time, had about thinking. Their question was, If you are awake, are you then busy with thinking? "No," one said. "I can walk in the woods and see the trees and all the other beautiful things, but I do not think I see the trees. I see ferns, and I see them without thinking." The other said, "Then you have been thinking, or you knew you were in the woods and that you saw trees, but only you did not use words."

I judged this controversy important and asked the opinion of Hans Freudenthal, a prominent Dutch mathematician and educator. His conclusion was clear: Thinking without words is not thinking. In *Structure and Insight* (van Hiele 1986), I expressed this point of view, and psychologists in the United States were not happy with it. They were right: If nonverbal thinking does not belong to real thinking, then even if we are awake, we do not think most of the time.

Nonverbal thinking is of special importance; all rational thinking has its roots in nonverbal thinking, and many decisions are made with only that kind of thought. We observe some things without having any words for them. We recognize the faces of familiar persons without being able to use words to describe their faces. In my levels of geometric thinking, the "lowest" is the **visual level,** which begins with nonverbal thinking. At the visual level of thinking, figures are judged by their appearance. We say, "It is a square. I know that it is one because I see it is." Children might say, "It is a rectangle because it looks like a box."

At the next level, **the descriptive level,** figures are the bearers of their properties. A figure is no longer judged because "it looks like one" but rather because it has certain properties. For example, an equilateral triangle has such properties as three sides; all sides equal; three equal angles; and symmetry, both about a line and rotational. At this level, language is important for describing shapes. However, at the descriptive level, properties are not yet logically ordered, so a triangle with equal sides is not necessarily one with equal angles.

At the next level, **the informal deduction level,** properties are logically ordered. They are deduced from one another; one property precedes or follows from another property. Students use properties that they already know to formulate definitions, for example, for squares, rectangles, and equilateral triangles, and use them to justify relationships, such as explaining why all squares are rectangles or why the sum of the angle measures of the angles of any triangle must be 180. However, at this level, the intrinsic meaning of deduction, that is, the role of axioms, definitions, theorems, and their converses, is not understood. My experience as a teacher of geometry convinces me that all too often, students have not yet achieved this level of informal deduction. Consequently, they are not successful in their study of the kind of geometry that Euclid created, which involves formal deduction. See van Hiele (1997) and Fuys, Geddes, and Tischler (1988) for further information about the levels.

How do students develop such thinking? I believe that development is more dependent on instruction than on age or biological maturation and that types of instructional experiences can foster, or impede, development. As I discuss at the end of this article, instruction intended to foster development from one level to the next should include sequences of activities, beginning with an exploratory phase, gradually building concepts and related language, and culminating in summary activities that help students integrate what they have learned into what they already know. The following activities illustrate this type of sequence for developing thinking at the visual level and for supporting a transition to the descriptive level.

Beginning Geometry and the Mosaic Puzzle

Join me now in using the seven-piece mosaic (see **fig. 1**) in playful explorations that deal with certain shapes and their properties, symmetry, parallelism, and area. Before reading further, please make your own set of pieces to use in the activities, which can be adapted for children, depending on their prior geometric experiences. **Figure 1** can be reproduced on cardstock to make durable sets for yourself and your students. Pieces are numbered on their topsides for reference in directions and discussions of the activities.

Imagine that the large rectangle in **figure 1** has broken into seven pieces: an isosceles triangle (piece 1); an equilateral triangle (piece 2); two right triangles (pieces 5 and 6); and three quadrilaterals consisting of a rectangle (piece 3), a trapezoid (piece 7), and an isosceles trapezoid (piece 4). **Figure 2** shows how the large rectangle and its pieces fit on a grid pattern of equilateral triangles.

We begin by asking, What can we do with these pieces? Children respond to this open question by using their imaginations and playing with the pieces

FIGURE 1

The seven-piece mosaic puzzle—create a set of pieces to use as you read this article.

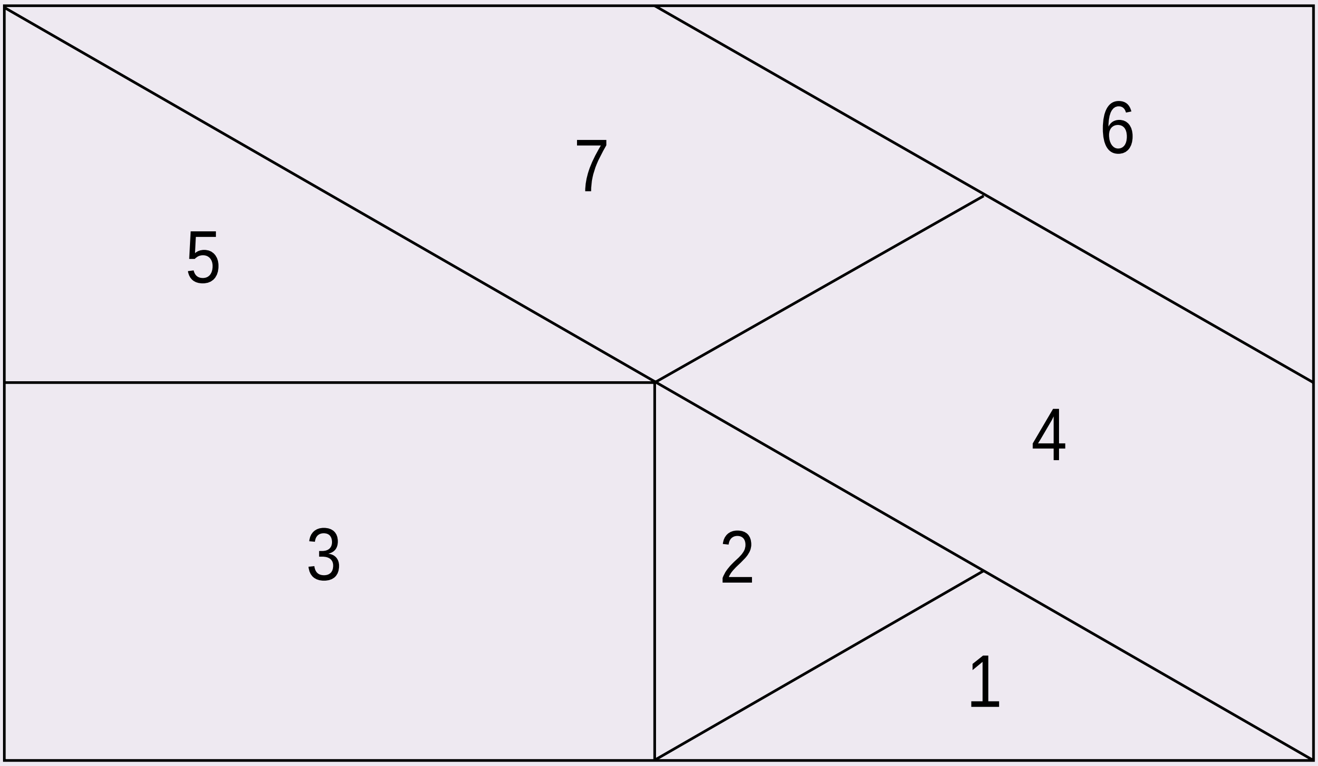


FIGURE 2

Equilateral-triangle grid

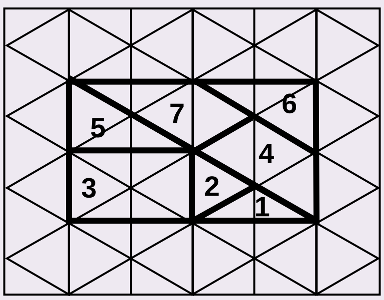
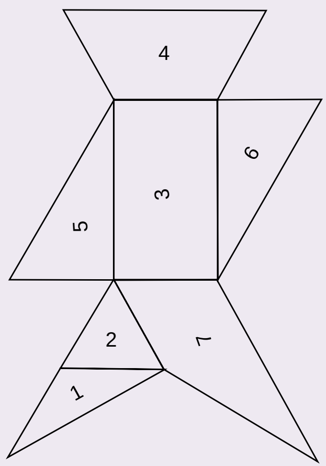


FIGURE 3

A person



to create whatever they wish—sometimes real-world objects like a person (see **fig. 3**) or a house (see **fig. 4**); sometimes other shapes, like piece 3, or abstract designs. Children should be given ample opportunity for free play and for sharing their creations. Such play gives teachers a chance to observe how children use the pieces and to assess informally how they think and talk about shapes.

In free play, children may have joined two pieces to make another piece, for example, using pieces 5 and 6 to make piece 3. We can ask them to find all the pieces that can be made from two others. Only pieces 1 and 2 cannot. Try this activity, and then find the one piece that can be made from three others. Children can place pieces directly on top of the piece that they want to make or form it next to the piece for easy visual comparison. To record their solutions, children should trace around a piece and then draw how they made it with the other pieces or show their method with colored markers on a triangle grid.

This activity leads children to notice that joining two pieces sometimes makes a shape that is not the same as one of the seven original pieces. They can investigate how many different shapes can be made with a pair of pieces, joining them by sides that match. With pieces 5 and 6, six shapes are possible, only one of which is the same as an original piece. Try these combinations, and then try the same activity with pieces 1 and 2.

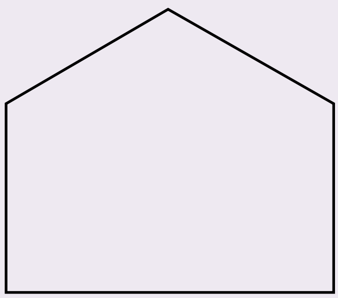
New shapes are introduced by puzzles that require two or more pieces. The shape in **figure 5** can be made two ways. One uses pieces 2 and 4 number side up; the other uses pieces 2 and 4 flipped over with the number side down. Make the shape both ways. Can it be made with pieces 1 and 7? With pieces 1 and 7 flipped over? What other two pieces make this shape, and do they also work if they are flipped over?

Making the shape in different ways with two pieces may inspire children to ask, Can we make it with three pieces, too? Try pieces 1, 2, and 5, and then make it in a different way with these three pieces. Also, try pieces 1, 2, and 5 flipped over.

In solving puzzles like these, children work visually with angles that fit and sides that match. They also notice that some pieces fit with either side up but that other pieces do not. Pieces 2 and 3 fit either side up; piece 7 does not, since flipping it changes its orientation and how it looks. Is piece 1 a flipable piece? Are pieces 4, 5, or 6?

FIGURE 4

A house



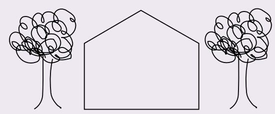
Puzzle Cards and the Mosaic Puzzle

Next I present more complex puzzles. Directions can be given orally or by task cards, like those in **figure 6.** Read and try them. They illustrate how puzzles that are created with two pieces can have solutions that use other pieces. Think about what geometry they involve and the conversations that

FIGURE 6

Task cards

House Puzzle
1. On a piece of paper, make a house like this one with two pieces.
2. Trace around the house you made to form a shape.
3. Make the shape with two other pieces.
4. Make the shape with three pieces. Can you find two ways to do it?
5. Can it be made with four pieces?



Tall-House Puzzle
1. On a piece of paper, make a tall house with piece 2 as the roof and one other piece.
2. Trace around the tall house you made.
3. Make the shape with pieces 5 and 7.
4. Can it be made with three pieces?

Make a Puzzle
1. Use any two, three, or four pieces. Make a shape. Trace around it on a large index card. Color it.
2. Can you make this shape with other pieces?
3. Write your name and a title for your shape on the index card.

FIGURE 5

This shape can be made in several different ways.

children might have while doing them.

Some students use strategies to solve these puzzles. For example, in part 4 of both house puzzles, children who know that rectangle piece 3 can be made from pieces 5 and 6 may use this relationship to figure out a solution by putting piece 1 or 2 on top and pieces 5 and 6 in the rectangular space on the bottom. It is important for children to share their approaches with classmates, perhaps by using an overhead projector to "show and tell." Teachers should also encourage problem posing. Children enjoy creating puzzles for others to solve. Puzzles can be presented as cutout shapes or can be drawn on cards and set out in a math center. Students can label puzzles with their names—for example, Big House Puzzle by Dina—which builds ownership of their creations.

Enlargements of pieces can be made; for example, pieces 2 and 4 make an enlargement of piece 2. Try this enlargement, and then make it with two other pieces, then with three. The enlargement has sides twice as long as piece 2, which we can readily see by making it on the triangle grid (see **fig. 7**). Using pieces 2, 4, 5, and 7, make an enlargement with sides three times as long as piece 2. Find four other pieces that work. Challenge: Make an enlargement with all seven pieces. Comparing the sides and angles of these triangles with those of piece 2, we see that the sides get progressively larger while the angles remain the same.

Exploring Geometry Shapes and Angles

Children soon notice that the sides of piece 2 have the same length, and likewise for the sides of each enlargement. So at this point, we can give a name for these figures—equilateral triangle—and ask students why the name is appropriate, that is, it has equal sides.

With this beginning, we can appreciate the advantages that this approach has for teaching geometry. First, children engage in activities that they think of as play and hence enjoy. They have puzzles to solve, and they learn things without the intention to learn. At appropriate times, teachers can introduce the names of pieces. After some time, children will use the names themselves and learn that the name remains the same no matter how the piece is placed. They also start to notice features of shapes. For example, piece 2 has equal sides; its corners are the same—are equal angles; and it looks the same when it is flipped—exhibits line symmetry—or turned—exhibits rotational symmetry. Children can learn about other pieces in a similar manner.

Next, the name *rectangle* is given for piece 3. Children are told that all three shapes in **figure 8** are rectangles, too, and asked to make them. Have children make the "tall" rectangle with pieces 1, 5, 6, and 7 and the rectangle in a "crooked" position with pieces 1 and 7 and the flip sides of pieces 5 and 6.

Can other rectangles be made? Of course, the largest is the large rectangle in **figure 1.** It is a challenge for students to reconstruct it without seeing the completed design. Children can arrange the pieces in several ways, and they enjoy finding new ways. By making various rectangles, children will—after some time—discover that all rectangles are not enlargements of one another, as was the case for equilateral triangles. Also, in contrast with equilateral triangles, the rectangle is a common everyday shape, and children should be asked to find and share examples of this shape from their home and school environments. After studying rectangles, children can investigate pieces 5 and 6, which form piece 3. These shapes are right triangles, or "rectangled triangles" as we call them in the Netherlands. Children can be asked to make other right triangles—for example, try 1, 2, 5, 6; or 3, 5, 6—and check whether they are all enlargements of piece 5.

Children can also play games that draw their attention to shapes and their parts. They could play "feel and find the shape," in which they hold a piece without seeing it and try to find the one that matches. Asking "How did you know?" encourages descriptive communication about the pieces, such as, "It has four sides and a pointy corner" for piece 7.

Fitting pieces into puzzles helps children become aware of the features of the sides and angles of the pieces. Some pieces have square corners, others have "sharp pointy" corners. Some have two equal

FIGURE 7

Enlargements of piece 2

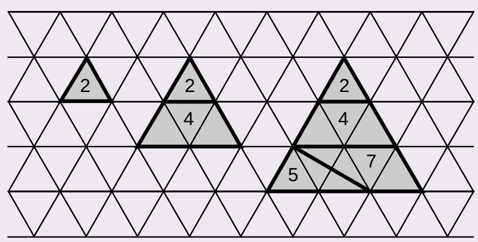
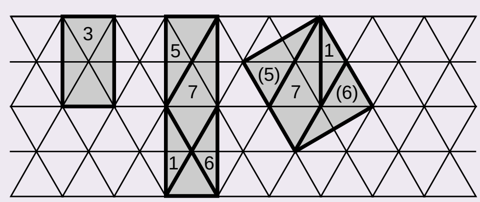


FIGURE 8

Rectangles



sides, whereas others have all equal sides or no equal sides. The language of sides and angles can now be introduced, but, of course, not with a formal definition. Students can compare triangle pieces and show how they are alike—for example, three sides, three angles—and different—for example, all sides equal, two sides equal, no sides equal, three angles the same. Piece 1 has two equal angles. What other piece has this property? Placing angles on top of each other to test whether they are equal helps children understand that the size of the angle is not dependent on the lengths of its sides.

Angles of the mosaic pieces come in five sizes. Asking children to compare angles of pieces with a square corner, or right angle, leads to informal work with acute angles—those smaller than a right angle—and with obtuse angles—those larger than a right angle. Building on the language that children invent for these kinds of angles, teachers can gradually introduce conventional terms. Children can find relationships between angles of pieces—for example, how the smallest angle relates to the other angles: they equal two, three, four, and five of the smallest. These activities are done without reference to angle measurement and build a foundation for later work with angles, their measurement in degrees, and angle relationships.

An interesting activity for children who know about angle measure is to figure out the measure of the angles in each of the seven pieces without using a protractor. Many ways are possible, and children should compare their approaches. Examine the pieces in **figure 1,** and find the measures of the angles of each piece. Think about the angle relationships that you used and whether you could figure out these measures in other ways by using other angle relationships.

Children who use the triangle grid to record solutions to puzzles become aware of equal angles in the grid and also of parallel lines. They can be asked to look for lines like train tracks and trace them with different-colored markers, creating designs that show three sets of parallel lines. Parallelism of lines is a feature needed for describing pieces 4 and 7—trapezoids, which have one pair of parallel sides—and also applies to the opposite sides of piece 3, a rectangle.

Other Activities with the Mosaic Puzzle

Placing pieces to fill in the space in puzzles also provides experiences with area. By direct comparison, students can show that some pieces take up more space than others—piece 7 has a greater area than piece 2—or can discover relationships, such as, piece 5 is half of piece 3. Working with shapes on the triangle grid reveals other relationships, such as, piece 4 has three times the area of piece 2, or how the area of piece 2 compares with the area of its enlargements (see **fig. 7**). A similar exploration of area could be done with piece 4 and its enlargements. These kinds of experiences with area lay a foundation for later work with square units of area and the discovery of ways to find the area of various shapes—for example, why the area of a right triangle is one-half the area of a rectangle—and how the enlargement of a shape, for instance, by doubling the lengths of its sides, affects its area.

To further develop children's descriptive thinking about the pieces, they can play "clue" games about the pieces or the shapes they made with them. Clues for piece 4 could be "four sides, four angles, two equal sides, two equal acute angles, and two parallel sides." Clues are revealed one at a time until the shape is identified. After each clue, children tell which pieces work or do not work and explain why. They could also play "guess the piece," in which they ask the teacher yes-no questions about the mystery shape. The teacher can list questions on the chalkboard and have children discuss whether all are needed to identify the shape. Children may point out that some properties imply others, such as "three sides" means that the shape has "three angles." These kinds of games give practice with properties that children have learned so far and strengthen children's use of descriptive language as a tool for reasoning about shapes and their properties. They also give teachers a window to children's developing levels of thinking, here between the descriptive level and the next level, where properties are logically ordered.

Having played with this special mosaic in these activities, we sense that many other questions to pose and topics to explore are possible. Furthermore, grids and mosaics based on other types of shapes can be used, such as one based on squares, leading in a natural way to area and to coordinate geometry, which connects shape and number.

Reflections on the Activities and Looking Ahead

Activities with mosaics and others using paper folding, drawing, and pattern blocks can enrich children's store of visual structures. They also develop a knowledge of shapes and their properties. To promote the transition from one level to the next, instruction should follow a five-phase sequence of activities.

Instruction should begin with an **inquiry phase** in which materials lead children to explore and discover certain structures. In the second phase,

direct orientation, tasks are presented in such a way that the characteristic structures appear gradually to the children, for example, through puzzles that reveal symmetry of pieces or through such games as "feel and find the shape." In the third phase, **explicitation,** the teacher introduces terminology and encourages children to use it in their conversations and written work about geometry. In a fourth phase, **free orientation,** the teacher presents tasks that can be completed in different ways and enables children to become more proficient with what they already know, for example, through explorations of making different shapes with various pieces or through playing clue games. In the fifth and final phase, **integration,** children are given opportunities to pull together what they have learned, perhaps by creating their own clue activities. Throughout these phases the teacher has various roles: planning tasks, directing children's attention to geometric qualities of shapes, introducing terminology and engaging children in discussions using these terms, and encouraging explanations and problem-solving approaches that make use of children's descriptive thinking about shapes. Cycling through these five phases with materials like the mosaic puzzle enables children to build a rich background in visual and descriptive thinking that involves various shapes and their properties.

Remember, geometry begins with play. Keep materials like the seven-piece mosaic handy. Play with them yourself. Reflect on what geometry topics they embody and how to sequence activities that develop children's levels of thinking about the topics. Then engage your students in play, activities, and games that offer an apprenticeship in geometric thinking. Children whose geometric thinking you nurture carefully will be better able to successfully study the kind of mathematics that Euclid created.

Watch for "Investigations: Are You Puzzled?" by Rosamond Welchman in the March 1999 issue for a related puzzle activity.

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BY WAY OF INTRODUCTION

(Continued from page 307)

would like to revise slightly Euclid's comment: "There is no royal road to teaching geometry"; it takes hard work by dedicated teachers to give students quality instruction. With this thought in mind, our hope is that you will find articles in this 1999 focus issue that will be useful in your journey along the road to accomplishing this worthy task.

Charles Geer
For the Editorial Panel

The Editorial Panel's commitment to features on geometry and geometric thinking goes beyond this focus issue. Be on the watch for future articles about this increasingly important mathematics topic, and please consider sharing your own ideas with the Panel. In particular, watch for the following articles that will be published in future issues of Teaching Children Mathematics: *"The Importance of Spatial Structuring in Geometric Reasoning," by Michael T. Battista; "Geometry and Op Art," by Evelyn J. Brewer; "Why Are Some Solids Perfect? Conjectures and Experiments by Third Graders," by Richard Lehrer and Carmen L. Curtis; and "Getting Students Actively Involved in Geometry," by Stuart P. Robertson.* ▲